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Abstract.

We consider inverse problems in Hilbert spaces contaminated by Gaussian noise whose covariance operator is not identity (i.e. it is not a white noise), and use a Bayesian approach to find its regularised smooth solution. We consider the so-called conjugate diagonal setting where the covariance operator of the noise and the covariance operator of the prior distribution are diagonal in the corresponding orthogonal bases of the Hilbert spaces defined by the forward operator of the inverse problem. Firstly, we derive the minimax rate of convergence in such problems with known covariance operator of the noise, showing that in the case of heterogeneous variance the ill-posed inverse problem can become self-regularised in some cases when the eigenvalues of the variance operator decay to zero, achieving parametric rate of convergence - as far as we are aware, this is a striking novel result that have not been observed before in nonparametric problems. Secondly, we give a general expression of the rate of contraction of the posterior distribution in case of known noise covariance operator in case the noise level is small, for a given prior distribution. We also investigate when this contraction rate coincides with the optimal rate in the minimax sense which is typically used as a benchmark for studying the posterior contraction rates. We apply our results to known variance operators with polynomially decreasing or increasing eigenvalues as an example. We also discuss when the plug-in estimator of the eigenvalues of the covariance operator of the noise does not affect the rate of the contraction of the posterior distribution of the signal. The Empirical Bayes estimator of prior smoothness proposed in [Knapik et al.(2012)] applies to our setting partially when the problem does not have the parametric rate of convergence. We also show that plugging in the maximum marginal likelihood estimator of the prior scaling parameter leads to the optimal posterior contraction rate, adaptively. Effect of the choice of the prior parameters on the contraction in such models is illustrated on simulated (synthetic) data with Volterra operator.

Some key words: Bayesian inference, inverse problems; rate of contraction; minimax rate

1. Introduction

Consider the following probability model for Y which are noisy indirect observations of unknown function μ:

$$Y = K\mu + \epsilon W \tag{1}$$

where $\mu \in H_1$, a separable Hilbert space and a known, injective, continuous, linear operator K maps μ into another separable Hilbert space, H_2. Here ϵ represents the level of noise, and W is a random process. For instance, under the white noise model, this model is equivalent to the setting of n observations from this model at n regularly spaced points with $\epsilon = \frac{1}{\sqrt{n}}$ [Brown and Low (1996)].

We consider ill-posed problems when the solutions even of a noise free problem (with $\epsilon = 0$) does not depend continuously on observations. This happens, for instance when the eigenvalues of operator K^*K decay to 0.

Typically, most methods for solving linear ill-posed inverse problems involve *regularising* the solution space, by constraining the set of solutions using some a priori information such as a small norm, sparsity or smoothness, normally leading to a unique solution in a noise free case. For further details, see [Engl(1996)]. Most regularised solutions can be interpreted as a Bayesian estimator where the regularisation is reflected as the prior information. For a more detailed discussion of correspondence between the penalised likelihood and Bayesian approaches, see [Bochkina(2013)].

However, the Bayesian perspective brings more than merely a different characterisation of a familiar numerical solution. Formulating a statistical inverse problem as one of inference in a Bayesian model has great appeal, notably for what this brings in terms of coherence, the interpretability of regularisation penalties, the integration of all uncertainties, and the principled way in which the set-up can be elaborated to encompass broader features of the context, such as measurement error, indirect observation, etc. The Bayesian formulation comes close to the way that most scientists intuitively regard the inferential task, and in principle allows the free use of subject knowledge in probabilistic model building [Rover et al.(2007), Voutilainen and Kaipio (2009), Kaipio and Fox (2011), Cotter et al.(2009), Auranen at al.(2005), etc]. For an interesting philosophical view on inverse problems, falsification, and the role of Bayesian argument, see [Tarantola(2013)]. Various Bayesian methods to solve inverse problems have been proposed [Wolpert and Ickstadt (2004), Efendiev et al.(2008), Kaipio et al.(1999), Cotter et al.(2009), Dashti et al.(2012)].

Solution to an inverse problem in the presence of noise is usually analysed by taking the limit of the noise $\epsilon \to 0$. In a Bayesian approach, the solution is a probability distribution over a set of functions which depends on observed data, making it a random probability distribution over a set of functions. [Ghosal and Ghosh (2000)] proposed to study *contraction rate* of the posterior distribution which is defined as the smallest $\varepsilon = \varepsilon(\epsilon)$ such that for every $M \to \infty$,

$$\mathbb{P}(\{\mu : ||\mu - \mu_0|| \geq M\varepsilon|Y\}) \stackrel{\mathbb{P}_{\mu_0}}{\to} 0 \text{ as } \epsilon \to 0 \tag{2}$$

uniformly over the true solution μ_0 in a relevant functional class (e.g. a Sobolev space).

In case of the inverse problem under the white noise model, the rate of contraction of the posterior distribution was studied by [Knapik et al.(2011)]. When the covariance operator of the noise is not a constant, this problem was studied by

[Agapiou et al.(2013)] and [Florens and Simoni (2016)], with the particular types of covariance operators motivated by the respective areas of application. Motivated by inverse problems arising in PDEs, [Agapiou et al.(2013)] considered a case where the covariance operators do not necessarily commute. In such challenging setting, to obtain the conditions for contraction of posterior distribution, the authors assumed the unknown function to be continuous, and expressed the smoothness of the unknown function in terms of the prior covariance operator which has restricted the range of applicability of their results in the setting where the operators commute; also, their contraction rates were slower than the minimax optimal ones in the case of the white noise [Knapik et al.(2011)], although in many cases the exponent in the rate could be arbitrarily close to the optimal exponent. [Florens and Simoni (2016)] investigated contraction of the posterior distribution in a challenging case of non-trivial covariance operators motivated by inverse problems arising in econometrics; to overcome the challenges, the authors assumed the covariance operator of the noise is trace class and true functions have monotonically decreasing coefficients in some basis (i.e. a subclass of Sobolev spaces) where they showed that the posterior contracts at the minimax rate, up to a log factor. [Agapiou and Mathe (2014)] proposed to choose a data-dependent prior mean as a way of making their posterior distribution adaptive and to contract at the minimax optimal rate; such approach is common in optimisation but it may be less appealing to a Bayesian statistician.

In this paper, we focus on the linear inverse problem when the covariance operator of the Gaussian noise W and operator KK^T are simultaneously diagonalisable but without any other constraints. We consider covariance operators that do not have to belong to the trace class, nor do we constrain our unknown function of interest to be continuous or have monotonically decreasing coefficients in some basis. Firstly, we derive the optimal rate in the minimax sense for this problem with known operator V over generalised Sobolev spaces which is typically used as a benchmark for studying the posterior contraction rates [Ghosal and Ghosh (2000)]. Secondly, we derive the rate of contraction of the posterior distribution of the unknown signal μ with known V and identify the prior distributions that lead to the contraction rate that coincides with the benchmark optimal rate in the minimax sense. We apply our results to known variance operators with polynomially decreasing or increasing eigenvalues as an example. As the covariance operator can be unknown, we investigate how using its plug-in estimator affects the rate. Effect of the choice of the prior parameters on the contraction in such models is illustrated on simulated (synthetic) data under Volterra operator.

One of the novel results we show is that in the heterogeneous case, when the eigenvalues of the variance operator decay to zero, the ill-posed inverse problem can become self-regularised to such a degree that the posterior distribution contracts at the parametric rate of convergence. We are not aware of such results known previously. [Agapiou et al.(2013)] state that in some cases regularisation is not necessary however their rate is not parametric; it corresponds to the nonparametric rate of convergence in the direct problem where $K = I$. We also demonstrate conditions when plugging in estimated eigenvalues of the covariance operator does not affect the rate. We also discuss when the plug-in estimator of the variance function does not affect the rate of the contraction of the posterior distribution.

The Empirical Bayes and Full Bayesian estimator of prior smoothness α proposed in [Knapik et al.(2012)] apply to our setting with polynomially decaying eigenvalues (when the problem does not have the parametric rate of convergence), with the same theoretical guarantees, i.e. producing an adaptive procedure that achieves the minimax rate of convergence.

In Section 2 we state the probability model, in Section 3 we state condition for posterior construction in general case. We discuss conclusions of these results in the case of a plug-in estimator of the covariance operator. In Section 4 we consider the case of polynomial eigenvalues of the covariance operator, deriving the minimax rate of convergence, the rate of contraction of the posterior distribution and the conditions on the prior making this rate minimax. In Section 7 effect of the choice of the prior parameters on the contraction rate is illustrated on simulated (synthetic) data. Proofs of the main general results are given in Section 8. We conclude with a discussion. The remaining proofs are given in appendix.

2. Bayesian model

In this paper we consider observational model (1) under assumption that $W = V^*Z \sim N(0, V^{*T}V^*)$, where Z is the isonormal process in H_2, $V^* : H_2 \mapsto H_2$ is a continuous linear operator. Denote $V = V^{*T}V^*$. Note that V, just like the identity element of H_2, does not have to be of trace class. Hence, (1) implies that $Y|\mu \sim N(K\mu, \epsilon^2 V^{*T}V^*)$.

We assume a Gaussian prior distribution:

$$\mu \sim N(0, \Lambda) \tag{3}$$

with covariance operator Λ belonging to a trace class ($tr(\Lambda) < \infty$).

Then, the posterior distribution of μ is given by

$$\mu \mid Y \sim N((K^TV^{-1}K + \epsilon^2\Lambda^{-1})^{-1}K^TV^{-1}Y, (\epsilon^{-2}K^TV^{-1}K + \Lambda^{-1})^{-1}).$$

Similar results were presented in [Agapiou et al.(2013)] and [Florens and Simoni (2016)]. The posterior distribution is proper if $trace((\epsilon^{-2}K^TV^{-1}K + \Lambda^{-1})^{-1}) < \infty$.

Our aim is to determine the contraction rate of the posterior distribution $\mu|Y$ around the true value μ_0 defined by (2). Below we consider $H_1 = H_2 = L^2$, and the corresponding norm $||\cdot||$ is the L^2 norm.

3. Sequence space model

3.1. Sequence space formulation

We reformulate the problem in the sequence space with respect to orthonormal bases of Hilbert spaces H_1 and H_2, in our case $H_1 = H_2 = L^2$. We are making the following assumptions.

Assumption 1 *(i) Operators $K^T K$ and Λ have the same eigenfunctions $\{e_i\}$, with eigenvalues $\{k_i^2\}$ and λ_i respectively.*

(ii) Operators KK^T and V have the same eigenfunctions $\{\phi_i\}$, with eigenvalues $\{k_i^2\}$ and σ_i^2 respectively.

Under this assumption, probability model (1) is equivalent to

$$Y_i \mid \mu_i \sim N(k_i\mu_i, \epsilon^2\sigma_i^2), \quad i = 1, 2, \dots, \text{ independently,} \tag{4}$$

with the prior model equivalent to

$$\mu_i \sim N(0, \lambda_i), \quad i = 1, 2, \dots, \text{ independently.}$$

Note that this distribution is proper if $\sum_i \lambda_i < \infty$.

Then, the corresponding posterior distribution of μ_i is

$$\mu_i | Y_i \sim N\left(\frac{Y_i k_i \lambda_i}{\lambda_i k_i^2 + \epsilon^2\sigma_i^2}, \frac{\sigma_i^2\lambda_i}{\epsilon^{-2}\lambda_i k_i^2 + \sigma_i^2}\right), \quad i = 1, 2, \dots, \text{ independently.} \tag{5}$$

In particular, if the prior is proper then the posterior is also proper. Note that the posterior distribution is from the same family of distributions as the prior, hence we refer to this case as the conjugate setting.

We assume that the inverse problem is mildly ill-posed:

Assumption 2 *We assume that eigenvalues (k_i^2) satisfy*

$$C_1^{-1} i^{-p} \le k_i \le C_1 i^{-p}$$

for some $p \ge 0, C_1 \ge 1$.

We consider the following generalised Sobolev class for the true unknown function μ_0.

Assumption 3 *We assume that the true unknown function μ_0 that generates the data according to the model (1) belongs to a generalised Sobolev class $S^\beta(A)$, $\beta, A > 0$:*

$$S^\beta(A) = \{f = \sum_i \phi_i f_i : \sum_i i^{2\beta} f_i^2 \le A^2\}.$$

We also make assumption about a priori smoothness of the unknown function.

Assumption 4 *We assume that eigenvalues (λ_i) satisfy*

$$\lambda_i = \tau_\epsilon^2 \, i^{-1-2\alpha}$$

for some $\alpha > 0$ and $\tau_\epsilon > 0$ such that $\epsilon^{-2}\tau_\epsilon^2 \to \infty$ as $\epsilon \to 0$.

This assumption implies that a priori we assume $\mu \in S^{\alpha'}$ almost surely for any $\alpha' < \alpha$ and a fixed ϵ.

3.2. Minimax rate of convergence

In this section we determine the minimax rate of convergence of estimators of μ under model (1).

The minimax risk in L^2 norm of estimator $\hat{\mu}$ of a true function μ_0 over Sobolev space $S^\beta(A)$ is given by

$$R(\hat{\mu}, S^\beta(A)) = \sup_{\mu_0 \in S^\beta(A)} \mathbb{E}_{\mu_0} ||\hat{\mu} - \mu_0||^2. \tag{6}$$

Definition 1 *ε_β is the minimax rate of convergence of estimators of μ under model (1) over $S^\beta(A)$ if $\exists 0 < c \le C < \infty$ that depend only on β and A such that*

$$c \le \inf_{\hat{\mu}} \varepsilon_\beta^{-2} R(\hat{\mu}, S^\beta(A)) \le C,$$

where $R(\hat{\mu}, S^\beta(A))$ is defined by (6).

This rate is usually used as a benchmark for posterior contraction rates [Ghosal and Ghosh (2000)].

The minimax rate of convergence of estimating μ in L^2 norm under model (1) over the generalised Sobolev class (or other smoothness classes) can be derived for given σ_i using Theorem 3 in [Belitser and Levit(1995)].

3.3. Rate of contraction of posterior distribution

Below we present a general result that can be applied for any λ_i, κ_i and σ_i satisfying stated conditions.

Theorem 1 *Under probability model (1) and prior (3) and Assumptions 1, for a monotonically increasing sequence $\sigma_i^2/[\lambda_i k_i^2]$, for every $M \to \infty$,*

$$\mathbb{P}(\{\mu : ||\mu - \mu_0|| \ge M\varepsilon | Y\}) \stackrel{\mathbb{P}_{\mu_0}}{\to} 0 \text{ as } \epsilon \to 0$$

uniformly over μ_0 in $S^\beta(A)$ where ε is given by

$$\varepsilon = \left[\epsilon^2 \sum_{i \le i_\epsilon} \sigma_i^2 k_i^{-2} + i_\epsilon^{-2\beta} + \sum_{i > i_\epsilon} \lambda_i + \epsilon^4 \max_{i \le i_\epsilon} \left[\frac{\sigma_i^2 i^{-\beta}}{k_i^2 \lambda_i} \right]^2 \right]^{1/2}$$

where $i_\epsilon = \max\{i : \sigma_i^2 k_i^{-2} \le \epsilon^{-2} \lambda_i\}$.

The first two terms represent variance and squared bias terms respectively, and the remaining terms involve prior parameters λ_i that can be chosen.

Note that this theorem easily generalises to the case of non-monotonic sequence $\lambda_i k_i^2/\sigma_i^2$, with the range $i > i_\epsilon$ becoming $I_\epsilon = \{i : \sigma_i^2/[\lambda_i k_i^2] > \epsilon^{-2}\}$.

Remark 1 *The posterior distribution can contract at parametric rate ϵ if (σ_i^2) is such that $\sum_{i=1}^\infty \sigma_i^2 k_i^{-2} \le C < \infty$, under the appropriate choice of prior parameters (λ_i). See Section 4 for details in the case of polynomially decaying σ_i.*

Remark 2 *There is a phenomenon known as saturation [Agapiou and Mathe (2014)] that constrains the posterior contraction rate for an undersmoothing prior. In the above theorem, this is described by the term* $\max_{i\le i_\epsilon}\left[\frac{\sigma_i^2 i^{-\beta}}{k_i^2\lambda_i}\right]$. *It will be illustrated in the examples below.*

Applying Theorem 1 to the case of mildly ill-posed problem and the unknown function being in a generalised Sobolev class, we get the following result.

Corollary 1 *Under assumptions of Theorem 1 and additional assumptions (2) and (4), for a monotonically increasing sequence* $\sigma_i^2 i^{2\alpha+1+2p}$*, the contraction rate* ε *of the posterior distribution is given by*

$$\varepsilon = \left[\epsilon\left(\sum_{i\le i_\epsilon}\sigma_i^2 i^{2p}\right)^{1/2} + i_\epsilon^{-\beta} + \tau_\epsilon i_\epsilon^{-\alpha} + \epsilon^2\tau_\epsilon^{-2}\max_{i\le i_\epsilon}\left[\sigma_i^2 i^{2p+2\alpha+1-\beta}\right]\right]$$

where $i_\epsilon = \max\{i : \sigma_i^2 i^{2\alpha+1+2p} \le \epsilon^{-2}\tau_\epsilon^2\}$.

4. Conjugate variance with polynomially decaying eigenvalues

4.1. Variance

In this section we consider a particular case of the variance decay.

Assumption 5 *Assume that eigenvalues* (σ_i^2) *of operator* V *satisfy*

$$C_2^{-1} i^\gamma \le \sigma_i \le C_2 i^\gamma$$

for some $\gamma\in\mathbb{R}$ *and* $C_2 \ge 1$.

4.2. Minimax rate of convergence

The minimax lower bound for model (1) is stated in the following proposition.

Proposition 1 *Under probability model (1) and Assumptions 1, 2 and 5, the minimax rate of convergence of estimating* μ *over the generalised Sobolev class* $S^\beta(A)$ *as defined by (1) is given by*

$$\varepsilon^* := \begin{cases} \epsilon^{\frac{2\beta}{1+2\beta+2(p+\gamma)}}, & \text{if } \gamma > -p-1/2.\\ \epsilon(\log|\epsilon|)^{1/2}, & \text{if } \gamma = -p-1/2.\\ \epsilon, & \text{if } \gamma < -p-1/2.\end{cases}$$

The proof is given in Section 8.

This result implies that for a heterogeneous variance the degree of ill-posedness for (1) changes, in particular to i.e. $\tilde p = p+\gamma$ if $p+\gamma > -1/2$. For $p+\gamma = 0$, the rate coincides with the minimax rate of the direct problem. If $p+\gamma \le -1/2$, the problem becomes self-regularised, i.e. the parametric rate of convergence ϵ can be achieved (up to a log factor in the case $p+\gamma = -1/2$).

Below is the intuitive derivation of the result when $\gamma \geq -p$. Since $\sigma_i > 0$ are known we can rescale the model (4) by σ_i:

$$\tilde{y}_i = \tilde{\kappa}_i \mu_i + \epsilon z_i$$

where $\tilde{y}_i := y_i/\sigma_i$ and $\tilde{\kappa}_i := \kappa_i/\sigma_i \asymp i^{-(p+\gamma)}$. The minimax rate of convergence under this model when $p+\gamma \geq 0$ is given in [Cavalier et al.(2002)]: $\epsilon^{\frac{2\beta}{1+2\beta+2\tilde{p}}}$ which coincides with the rate stated in Proposition 1 in this case.

4.3. Contraction Rates

Having found the minimax rates, we can now discuss the contraction rates achieved by the posterior distribution under the considered Bayesian model. Note these rates also apply when the problem is self-regularised, i.e. $p + \gamma \leq -1/2$.

Theorem 2 *Under probability model (1), prior (3) and assumptions 1, 2, 3, 4, 5, for every $M \to \infty$,*

$$\mathbb{E}_{\mu_0}\mathbb{P}(\{\mu : ||\mu - \mu_0|| \geq M\varepsilon | Y\}) \to 0 \text{ as } \epsilon \to 0$$

where

$$\varepsilon := \begin{cases} (\epsilon^2\tau_\epsilon^{-2})^{\frac{\beta}{1+2\alpha+2(p+\gamma)}\wedge 1} + \tau_\epsilon(\epsilon^2\tau_\epsilon^{-2})^{\frac{\alpha}{1+2\alpha+2(p+\gamma)}}, & \text{if } \gamma > -p-1/2, \\ (\epsilon^2\tau_\epsilon^{-2})^{\frac{\beta}{2\alpha}\wedge 1} + \epsilon[\log(\epsilon^{-1}\tau_\epsilon)]^{1/2}, & \text{if } \gamma = -p-1/2, \\ (\epsilon^2\tau_\epsilon^{-2})^{\frac{\beta}{1+2\alpha+2(p+\gamma)}\wedge 1} + \epsilon, & \text{if } -p-1/2-\alpha < \gamma < -p-1/2 \end{cases}$$

uniformly over μ_0 in $S^\beta(A)$.

The assumption of monotonicity of $\sigma_i^2/[\lambda_i k_i^2] \asymp i^{1+2\gamma+2\alpha+2p}$ for large i from Theorem 1 is reflected in the condition $-p-1/2-\alpha < \gamma$.

Recall that parameters p and γ are assumed known and given by the problem, as well as the smoothness parameter β. Parameters of the prior α and τ_ϵ can be chosen in some cases so that the posterior contracts at the optimal rate given in Proposition 1.

Corollary 2 *Let assumptions of Theorem 2 hold. The rate of contraction of the posterior given in Theorem 2 matches the minimax rate of convergence, up to a constant, for the following α and τ_ϵ:*

(i) $\tau_\epsilon = const \in (0,\infty)$ *and*

$$\begin{cases} \alpha = \beta, & \text{if } \gamma > -\frac{1+2p}{2}, \\ \alpha \leq \beta, & \text{if } \gamma = -\frac{1+2p}{2}, \\ \alpha \leq \beta, & \text{if } -\frac{1+2p}{2} - \alpha < \gamma < -\frac{1+2p}{2}. \end{cases}$$

(ii) for τ_ϵ depending on ϵ:

$$\begin{cases} \alpha \geq \beta/2 - (1/2+p+\gamma), \quad \tau_\epsilon = C\epsilon^{\frac{2(\beta-\alpha)}{1+2\beta+2(p+\gamma)}}, & \text{if } \gamma > -\frac{1+2p}{2}, \\ \epsilon^{-B} \geq \tau_\epsilon \geq C\epsilon^{1-\max(1/2,\alpha/\beta)}[\log \epsilon^{-1}]^{-0.5\max(1/2,\alpha/\beta)}, & \text{if } \gamma = -\frac{1+2p}{2}, \\ \tau_\epsilon \geq C\epsilon^{\min(1/2, 1-\frac{1+2\alpha+2(p+\gamma)}{2\beta})}, & \text{if } -\frac{1+2p}{2} - \alpha < \gamma < -\frac{1+2p}{2}. \end{cases}$$

Observe that when $\gamma > -p-1/2$, the rates obtained are similar to the homogeneous case, albeit with a different degree of ill-posedness $\tilde{p} = p + \gamma$. When $p + \gamma < 0$, the fastest rate of contraction coincides with the minimax rate of convergence of the direct problem, i.e. the model self-regularises, and it can achieved when we undersmooth a priori.

If $\alpha < \beta/2 - (1/2 + p + \gamma)$ and $\gamma > -p - 1/2$, i.e. if we undersmooth too much a priori, then the minimax rate cannot be achieved for any τ_ϵ. When $\gamma = 0$, this coincides with the findings of [[Knapik et al.(2011)]]. This is known as saturation [Agapiou and Mathe (2014)]. However, in the self-regularising case $-p-1/2-\alpha < \gamma \leq -p-1/2$ the optimal rate can be achieved if the appropriate prior scaling is used.

Therefore, in principle, in some cases of the considered heterogeneous variance model it is possible for the posterior distribution to contract at the parametric rate of convergence, which is not possible in the case of the white noise model.

Remark 3 *Note that the case $p + \gamma + 1/2 > 0$, for a given $\alpha \geq \beta/2 - (1/2 + p + \gamma)$, $\alpha > 0$, the value of τ_ϵ that leads to the minimax rate is such that the cutoff level*

$$i_\epsilon \asymp [\epsilon^{-2}]^{\frac{1}{1+2\beta+2(p+\gamma)}}$$

is independent of α and is the cutoff level corresponding to the minimax optimal projection estimator (projecting on the first i_ϵ components).

Note [Florens and Simoni (2016)] consider the case of our setup with $p+\gamma > 0$ and $|\mu_{0,i}| \asymp i^{-\beta-1/2}$.

In [Agapiou et al.(2013)], the rate of contraction in this setting is given only for $\beta > \alpha + 1/2$ (in our notation) by

$$\epsilon^{\frac{\beta \wedge (p+\gamma+2\alpha+1)}{\delta(1+2\alpha)+1+2p+2\gamma+2[\beta \wedge (p+\gamma+2\alpha+1)]}}, \quad \forall \delta > 0$$

(where $\gamma_A = \beta/(\alpha+1/2)$, $\ell_A = p/(2\alpha+1)$, $s_{0,A} = 1/(2\alpha+1)$, $\Delta_A = (p+\gamma)/(\alpha+1/2)+1$, with subscript A referring to parameters in [Agapiou et al.(2013)]). In particular, the authors' assumption $\Delta_A > 2s_{0,A}$ is equivalent to assumption $p+\gamma+\alpha+1/2 > 1$ in our notation which is stronger than our assumption $p+\gamma+\alpha+1/2 > 0$. In fact, under the latter assumption, it is not possible to achieve the minimax optimal contraction rate for $p + \gamma > -1/2$ with constant τ_ϵ which is achieved with $\alpha = \beta$. Also, the authors refer to the case $p + \gamma < 0$ as self-regularising, with no regularisation being necessary which we do not find in case $p + \gamma \in (-1/2, 0)$; also their rate in this case is not parametric unless $\delta(1/2 + \alpha) = -(1/2 + p + \gamma) - 0.5\beta \wedge (p + \gamma + 2\alpha + 1)$ which is possible only if $\alpha < -1.5(p + \gamma + 1/2) - 1/4$ and $\beta < -2(1/2 + p + \gamma)$.

5. Contraction rates with plug-in estimator of $V(x)$

5.1. General case

When the covariance operator V is unknown, sometimes its plug-in estimator, and hence of σ_is, is used to conduct inference about μ. We investigate how this affects the contraction rate of the posterior distribution of μ. Theorem 1 (and Corollary 1 for polynomial α_i and k_i) can be used address this question.

Suppose we have a plug-in estimator of operator V, $\hat{V}$, which under Assumption 1 has fixed eigen functions $\{\phi_i\}$ and estimated eigenvalues $\{\hat{\sigma}_i^2\}$. We consider the case where $\hat{\sigma}_i^2$ are independent of Y, and their mean square error is bounded by ϵ_σ^2. Plugging in an estimator can be thought of as having an informative prior distribution on V that is a point mass at $\hat{V}$.

Assumption 6 *Assume that the estimated eigenvalues ($\{\hat{\sigma}_i^2\}$) of operator V are independent of Y, and there exists a constant c_0 such that*

$$P(|\hat{\sigma}_i^2 - \sigma_i^2| \leq c_0 \epsilon_\sigma,\ i = 1, 2, \ldots) \to 1 \text{ as } \epsilon_\sigma \to 0.$$

These assumptions are satisfied when variances are estimated from a different study, or when there are repeated observations, and under the Gaussian error, the sample mean and the sample variance are independent. We illustrate how this applies to the case of repeated observations in Section 5.3.

As in the considered case $\sigma_i \to 0$, we propose to truncate the estimator of σ_i^2 and use

$$\tilde{\sigma}_i^2 = \max(c_0 \epsilon_\sigma, \hat{\sigma}_i^2).$$

Plugging-in estimators for σ_i^2, the posterior distribution of μ_i (5) becomes

$$p(\mu_i \mid Y_i, \tilde{\sigma}_i^2) \approx N\left(\frac{Y_i k_i \lambda_i}{\lambda_i k_i^2 + \epsilon^2 \tilde{\sigma}_i^2}, \frac{\tilde{\sigma}_i^2 \lambda_i}{\epsilon^{-2} \lambda_i k_i^2 + \tilde{\sigma}_i^2}\right) \qquad i = 1, 2, \ldots, \text{ independently.}$$

In particular, Theorem 1 and Corollary 1 imply that when $\sigma_i \geq c_1 > 0$ for all i, then using a consistent plug-in estimator (i.e. when plugged in values are $\hat{\sigma}_i^2 = \sigma_i^2 + o(1)$), the effect on the contraction rates of the posterior distribution is a larger constant on the definition of the summation set. When sequence (σ_i) decreases to 0 as i increases then the error of estimation of V can affect the rate, with the effect depending on the speed of decay of (σ_i).

We illustrate this application on the example of a plug-in estimator when the unknown true values of σ_i^2 satisfy (5), and the plug in estimator of σ_i^2 has an error bound ϵ_σ. We only consider the case $\gamma < 0$, as otherwise $i^{2\gamma}$ dominates ϵ_σ, and the error of the plug-in estimator does not affect the rate.

We investigate how the contraction rates are affected as $\epsilon_\sigma \to 0$.

Theorem 3 *Consider probability model (1), prior distribution (3) and Assumptions 1, 3 with monotonically increasing sequence $\sigma_i^2/[\lambda_i k_i^2]$. Consider an estimator of $\{\sigma_i^2\}$ satisfying Assumption 6, and use a plugin estimator $\tilde{\sigma}_i^2 = \max(c_0\epsilon_\sigma, \hat{\sigma}_i^2)$.*

Then, for every $M \to \infty$,

$$\mathbb{P}(\{\mu : ||\mu - \mu_0|| \geq M \varepsilon_{plugin} | Y, \tilde{\sigma}_i\}) \overset{\mathbb{P}_{\mu_0, V}}{\to} \text{ as } \epsilon \to 0$$

uniformly over μ_0 in $S^\beta(A)$ where ε is given by

$$\begin{aligned}\varepsilon^2_{plugin} &= \epsilon^2 \sum_{i\in \bar I_\epsilon(2)\cup \bar I_\sigma(2)} \sigma_i^2 k_i^{-2} + \epsilon^2 c_0\epsilon_\sigma \sum_{i\in I_\sigma(2)\cap \bar I_{\sigma\epsilon}(1/3)} k_i^{-2}\\ &+ \sum_{i\in I_\epsilon(2/3)} \lambda_i + \sum_{i\in I_\sigma(2)\cap I_{\sigma\epsilon}(1/3)} \lambda_i\\ &+\epsilon^4 \max\left\{\max_{i\in \bar I_\epsilon(1)}\left[\frac{\sigma_i^2 i^{-\beta}}{k_i^2\lambda_i}\right]^2, \max_{i\in \bar I_{\sigma\epsilon}(1)}\left[\frac{\epsilon_\sigma i^{-\beta}}{k_i^2\lambda_i}\right]^2\right\} + \max_{i\in I_\epsilon(1)\cap I_{\sigma\epsilon}(1)} i^{-2\beta}\end{aligned}$$

where

$$I_\epsilon(a) = \{i:\, \sigma_i^2/[\lambda_i k_i^2] > a\epsilon^{-2}\}, \tag{7}$$

$$I_\sigma(a) = \{i:\, \sigma_i^2 < ac_0\epsilon_\sigma\}, \tag{8}$$

$$I_{\sigma\epsilon}(a) = \{i:\, c_0\epsilon_\sigma/[\lambda_i k_i^2] > a\epsilon^{-2}\}. \tag{9}$$

In particular, if $I_\sigma(2/3) \subseteq I_\epsilon(2)$ then the rate consists of the same terms as the contraction rate for the known V, with slightly larger constant in I_ϵ in some terms:

$$\varepsilon^2_{plugin} = \epsilon^2 \sum_{i\in \bar I_\epsilon(2)} \sigma_i^2 k_i^{-2} + \sum_{i\in I_\epsilon(2)}\lambda_i + \epsilon^4 \max_{i\in \bar I_\epsilon(1)}\left[\frac{\sigma_i^2 i^{-\beta}}{k_i^2\lambda_i}\right]^2 + \max_{i\in I_\epsilon(1)} i^{-2\beta}.$$

In the next section we show how this works for polynomially decaying σ_i.

5.2. Polynomially decaying eigenvalues

In this section, we investigate how a plug-in estimator of $V(x)$ affects the rate of contraction under the setting of Section 4 using Theorem 3.

Theorem 4 *Consider probability model (1) with prior (3) under Assumptions 1, 2, 4, 5 with $\gamma < 0$. Assume that a plug-in estimator $\hat V$ of variance operator V satisfies assumption 6.*

(i) If $\epsilon_\sigma < C[\epsilon^2\tau_\epsilon^{-2}]^{-\gamma/(\alpha+1/2+p+\gamma)}$ then the rate of contraction is not affected by using a plug-in estimator of V, i.e. it is coincides with the rate given in Theorem 2, up to a constant.

(ii) If $\epsilon_\sigma \geq C[\epsilon^2\tau_\epsilon^{-2}]^{-\gamma/(\alpha+1/2+p+\gamma)}$, then the contraction rate of the posterior distribution is given by

$$\begin{aligned}\varepsilon^2_{plugin} &= \epsilon^2(\log\epsilon_\sigma^{-1})^{\mathbb{I}\{1+2(p+\gamma)=0\}} + \tau_\epsilon^2[\epsilon_\sigma^{-1}\epsilon^{-2}\tau_\epsilon^2]^{-2\alpha/(1+2\alpha+2p)} + [\epsilon_\sigma^{-1}\epsilon^{-2}\tau_\epsilon^2]^{-2\beta/(1+2\alpha+2p)}\\ &+\epsilon^4\tau_\epsilon^{-4}\epsilon_\sigma^{\frac{(1+2\alpha+2(p+\gamma)-\beta)_+}{\gamma}}.\end{aligned}$$

That is, if the error ϵ_σ of estimating (σ_i^2) is small enough, then the rate of estimation of μ is not affected.

In the next section we shall see how this applies to the case of repeated observations.

5.3. Example with repeated observations

Now we suppose that we have m independent replicates of the original model (4):

$$Y_{i,j} \sim N(k_i\mu_i, \epsilon_0^2\sigma_i^2), \quad i = 1, \dots, n \ \text{ and } j = 1, \dots, m, \tag{10}$$

independently. Consequently, for each i, the sample mean ($\bar{Y}_i$) and the sample variance (s_i^2) are defined as follows

$$\bar{Y}_i := \frac{1}{m}\sum_{j=1}^{m} Y_{i,j}, \quad \text{and } s_i^2 := \frac{1}{m-1}\sum_{j=1}^{m}(Y_{i,j} - \bar{Y}_i)^2.$$

By the properties of the normal distribution, $\frac{m-1}{\sigma_i^2}s_i^2 \sim \chi^2_{m-1}$ independently for all i, and independently of $\bar{Y}_i$. Then,

$$\bar{Y}_i \sim N(k_i\mu_i, \epsilon^2\sigma_i^2), \quad i = 1, \dots, n \ \text{ and } j = 1, \dots, m, \tag{11}$$

where $\epsilon^2 = \epsilon_0^2/m$.

Now we study when simultaneous asymptotic consistency holds for large m with high probability for the following estimator of $(\sigma_i^2)_{i\geq 1}$, given a positive integer M:

$$\hat{\sigma}_i^2 = s_i^2 I(i \leq M), \quad i \geq 1. \tag{12}$$

Proposition 2 *Assume probability model (10), and consider the truncated estimator of $(\sigma_i^2)_{i\geq 1}$ defined by (12). Fix $a \in \mathbb{R}^+$, and define*

$$M_\sigma = M_\sigma(\epsilon_\sigma) = \inf_{M\in\mathbb{N}}\{\sigma_i^2 \leq c_0\epsilon_\sigma, \ \forall i > M\}. \tag{13}$$

Then, $\forall M \geq M_\sigma$ and ϵ_σ such that $c_0\epsilon_\sigma/c_\sigma \leq 1/2$,

$$P(|\hat{\sigma}_i^2 - \sigma_i^2| \leq c_0\epsilon_\sigma, \ \forall i \geq 1) \geq 1 - 2Me^{-(m-1)(c_0\epsilon_\sigma/c_\sigma)^2/6}.$$

Note that assumption $M > M_\sigma$ ensures $\{i : i > M\} \subseteq I_\sigma(1) \subset I_\sigma(2/3)$.

Consequently, we obtain the following corollary.

Corollary 3 *Under the assumptions stated in Proposition 2, $P(|\hat{\sigma}_i^2 - \sigma_i^2| \leq c_0\epsilon_\sigma, \ i = 1, \dots, M) \to 1$ if $m\epsilon_\sigma^2 \to \infty$ and $\frac{\log M}{m\epsilon_\sigma^2} \to 0$ as $m \to \infty$.*

Therefore, under the assumptions stated in Theorem 3 adapted to the model of Proposition 2, we obtain the following result.

Theorem 5 *Assume probability model (10) and a prior distribution on μ (3) under Assumptions 1, 3, and consider the truncated estimator of $(\sigma_i^2)_{i\geq 1}$ defined by (12) with M*

$$m\epsilon_\sigma^2 \to \infty \quad \textit{and} \quad \frac{\log M}{m\epsilon_\sigma^2} \to 0$$

and $M \geq M_\sigma(\epsilon_\sigma)$ where $M_\sigma(\epsilon_\sigma)$ is defined by Equation (13).

Then, for every $M_0 \to \infty$,

$$\mathbb{P}(\{\mu : ||\mu - \mu_0|| \geq M_0\varepsilon_{plugin}|Y, (\tilde{\sigma}_i^2)\}) \overset{\mathbb{P}_{\mu_0,V}}{\to} \quad \textit{as } \epsilon \to 0$$

uniformly over μ_0 in $S^\beta(A)$ where ε_{plugin} is the rate given in Theorem 3.

Now consider the case where the eigenvalues of the variance operator satisfy Assumption 5. Under the assumptions stated in Proposition 2, it is easy to see that $M_\sigma = (\frac{c_0\epsilon_\sigma}{C_2})^{\frac{1}{2\gamma}}$. Also, for this model, given M such that $\frac{\log M}{m} \to 0$, the rate of convergence ϵ_σ satisfies the following conditions:

$$\epsilon_\sigma^{-1}\sqrt{\frac{\log M}{m}} = o(1) \quad and \quad \epsilon_\sigma \geq \frac{C_2}{c_0}M^{2\gamma},$$

with the smallest ϵ_σ being $\epsilon_\sigma = \frac{C_2}{c_0}M^{2\gamma}$. In this case, M must satisfy $M^{-4\gamma}\log M = o(m)$ which holds if $M = o\left(\left[\frac{m}{\log m}\right]^{1/(4|\gamma|)}\right)$. This results in the rate of convergence satisfying $\epsilon_\sigma\left[\frac{m}{\log m}\right]^{1/2} \to \infty$ which usually holds for nonparametric models.

If the eigenvalues of the variance operator satisfy Assumption 5, then for any $M = o\left(\left[\frac{m}{\log m}\right]^{1/(4|\gamma|)}\right) \to \infty$, the corresponding rate of convergence is $\epsilon_\sigma = \frac{C_2}{c_0}M^{2\gamma}$. In this case, according to Theorem 4, the rate of contraction of the posterior is not affected by the plug-in if $\epsilon_\sigma = \frac{C_2}{c_0}M^{2\gamma} \leq C[\epsilon^2\tau_\epsilon^{-2}]^{-\gamma/(\alpha+1/2+p+\gamma)}$.

For $\epsilon_0 = 1$, the rate of contraction of the posterior is not affected by the plug-in estimator if

$$M \geq C(m^{0.5}\tau_\epsilon)^{1/(\alpha+1/2+p+\gamma)} \quad and \quad M = o\left(\left[\frac{m}{\log m}\right]^{1/(4|\gamma|)}\right) \to \infty.$$

First consider the case $\tau_\epsilon = const$. Then we must have

$$M = o\left(\left[\frac{m}{\log m}\right]^{1/(4|\gamma|)}\right) \geq Cm^{0.5/(\alpha+1/2+p+\gamma)},$$

i.e. $\alpha > -3\gamma - (1/2+p)$.

For the contraction rate of the posterior of μ with $\tau_\epsilon = 1$ to be optimal, we need to have $\alpha = \beta$ if $\gamma > -p - 1/2$, and $\alpha \leq \beta$ if $\gamma \leq -p - 1/2$ (Corollary 2). This is possible with the plug-in estimator of σ_i if $\beta > (-(1/2+p) - 3\gamma)_+$, i.e. if $\gamma > -(1/2+p)/3$, or $\gamma \leq -(1/2+p)/3$ and $\beta > -(1/2+p) - 3\gamma$ when the contraction rate becomes slower.

For τ_ϵ that can depend on ϵ, Corollary 2 gives us the following conditions for the contraction rate to be optimal.

(i) $\gamma > -\frac{1+2p}{2}$: $\tau_\epsilon = C\epsilon^{\frac{2(\beta-\alpha)}{1+2\beta+2(p+\gamma)}}$ and $\alpha \geq \beta/2 - (1/2+p+\gamma)$.
The constraint on M becomes $M \geq Cm^{0.5/(1/2+\beta+p+\gamma)}$. This M satisfies condition $M = o\left(\left[\frac{m}{\log m}\right]^{1/(4|\gamma|)}\right)$ if

$$m^{(1+2\beta+2p+6\gamma)/[4\gamma(1+2\beta+2p+2\gamma)]}[\log m]^{-1/(4\gamma)} = o(1)$$

i.e. if $1+2\beta+2p+6\gamma > 0$. Otherwise, the rate of contraction is suboptimal.

Constraints	$\tau_\epsilon = const$	$\tau_\epsilon \neq const$ (= optimal value)
$1/2+p+\gamma \geq -2\gamma$	any $\beta > 0$	any $\beta > 0$
$0 < \gamma+p+1/2 < -2\gamma$	$\beta > -1/2-p-3\gamma$	$\beta > -1/2-p-3\gamma$
$\gamma+p+1/2 = 0$	$\beta > -2\gamma$	$\beta > -2\gamma$
$-\alpha < \gamma+p+1/2 < 0$	$\beta > (-1/2-p-3\gamma)_+$	$\beta > -\gamma$

Table 1. Repeated observations, $\sigma_i \asymp i^\gamma$, $\gamma < 0$: when the optimal contraction rate of the posterior with the plug-in estimator of (σ_i) is not affected by the plug-in. Constraints on τ_ϵ and α are as given in Corollary 2, and $\alpha > -\gamma$ if $\gamma + p + 1/2 = 0$ and $\tau_\epsilon \neq const$.

(ii) Case $\gamma = -\frac{1+2p}{2}$: $\epsilon^{-B} \geq \tau_\epsilon \geq C\epsilon^{1-\max(1/2,\alpha/\beta)}[\log \epsilon^{-1}]^{0.5\max(1/2,\alpha/\beta)}$.
The constraint on M becomes $M \geq C[m\log m]^{\max(1/(4\alpha),1/(2\beta))}$. It is possible to satisfy condition $M = o\left(\left[\frac{m}{\log m}\right]^{1/(4|\gamma|)}\right)$ if $|\gamma| < \min(\alpha, \beta/2)$, otherwise the rate is suboptimal.

(iii) Case $-\frac{1+2p}{2} - \alpha < \gamma < -\frac{1+2p}{2}$: $\tau_\epsilon = C\epsilon^{\min(1/2, 1-\frac{1+2\alpha+2(p+\gamma)}{2\beta})}$. The constraint on M becomes $M \geq Cm^{1/[4\min(\alpha+1/2+p+\gamma,\beta)]}$. This M satisfies condition $M = o\left(\left[\frac{m}{\log m}\right]^{1/(4|\gamma|)}\right)$ if

$$m^{1/[4\min(\alpha+1/2+p+\gamma,\beta)]+1/(4\gamma)}[\log m]^{-1/(4\gamma)} = o(1),$$

i.e. if $\beta + \gamma > 0$. Otherwise, the rate is suboptimal.

These conclusions are summarised in Table 1. Only in the case $1/2 + p \geq -3\gamma$ the optimal rate is not affected by the plug-in estimator for any $\beta > 0$; for $1/2 + p < -3\gamma$, for small β the rate is no longer optimal.

6. Empirical Bayes estimator

Consider the prior Gaussian distribution with $\lambda_i = \tau i^{-2\alpha-1}$, with empirical Bayes posterior of μ using the marginal likelihood estimator of τ and fixed α:

$$\hat{\tau} = \arg\max_{\tau>0} \int p(y \mid \mu, \sigma) dP(\mu \mid \tau, \alpha) \tag{14}$$

$$= \arg\min_{\tau>0} \sum_{i=1}^{\infty} \left[\frac{y_i^2}{k_i^2\lambda_{0,i}\tau + \epsilon^2\sigma_i^2} + \log(k_i^2\lambda_{0,i}\tau + \epsilon^2\sigma_i^2)\right] \tag{15}$$

where $\lambda_{0,i} = i^{-2\alpha-1}$.

For $p + \gamma > 0$, after rescaling y_i by σ_i, the approach becomes that of an inverse problem under white noise considered by [Knapik et al.(2012)] who constructed an empirical Bayes estimator (and the adaptive full posterior) that leads to the optimal posterior contraction rates for μ. However, as far as we can see, their proof does not work in the case $p + \gamma < 0$. Case $p + \gamma = 0$, after rescaling of y_i by σ_i, corresponds to the white noise direct model.

[Florens and Simoni (2016)] constructed an empirical Bayes estimator of τ with Gamma prior that does apply to the case $p+\gamma<0$ however they showed that it led to suboptimal posterior contraction rate of μ.

We prove that the posterior distribution of μ with plugged in value of $\hat{\tau}$ (14) contracts at the optimal rate adaptively for $0<\beta\le B_0<\infty$, in the minimax sense. An alternative way of proving concentration of the Empirical Bayes posterior is to verify the assumptions of Theorem 2.1 in [Rousseau and Szabo (2017)], however the conditions are general and hence are relatively complicated. The authors verified these conditions for our model with $p=\gamma=0$ under an additional assumption on μ_0 that it also belongs to a Hölder space with the same exponent β). The proof we give for the considered model does not require additional constraints on μ_0, and it also gives insight in the behaviour of the EB estimate $\hat{\tau}$.

Theorem 6 *Consider probability model (1) and prior (3) under Assumptions 1, 2, 3, 4, 5. Denote* $\tau=\tau_\epsilon^2$.

Then, if $\alpha>(-p-\gamma-1/2)_+$ *and* $\beta+\alpha+1+2\tilde{p}>0$ *(achieved e.g. if* $\alpha\ge-2p-2\gamma-1/2$*), the solution of (14),* $\hat{\tau}$*, satisfies the following equation with probability tending to 1 as* $\epsilon\to0$*:*

$$\hat{\tau}\asymp\begin{cases}\epsilon^{-4(\alpha-\beta)/(1+2p+2\gamma+2\beta)}(1+o_P(1)), & \text{if } \alpha+1/2\ge\beta,\\ \epsilon^{1/(1+p+\gamma+\alpha)}(1+o_P(1)), & \text{if } \alpha+1/2<\beta.\end{cases}\tag{16}$$

Under the above conditions, the posterior contraction rate of μ *with plugged in* $\tau_\epsilon=\sqrt{\hat{\tau}}$ *is given by, for every* $M\to\infty$*,*

$$\mathbb{E}_{\mu_0}\mathbb{P}(\{\mu:\,||\mu-\mu_0||\ge M\varepsilon|Y,\hat{\tau}\})\to0\ \text{as}\ \epsilon\to0$$

uniformly over μ_0 *in* $S^\beta(A)$*, where*

$$\varepsilon=\epsilon^{2[\beta\wedge(\alpha+1/2)\wedge(1+2\alpha+2\tilde{p})]/(1+2\tilde{p}+2(\beta\wedge(\alpha+1/2)))}+\epsilon[\log\epsilon^{-1}]^{0.5I(p+\gamma=-1/2)}.$$

Therefore, posterior distribution with the plugged in estimator $\hat{\tau}$ *is consistent. The contraction rate is optimal in the minimax sense adaptively over* $S^\beta(A)$ *for* $0<\beta\le B_0<\infty$ *as long as*

$$\alpha\ge\max(B_0/2-1/2+(-p-\gamma)_+,(-p-\gamma)_+,-2(p+\gamma)-1/2).\tag{17}$$

The theorem states that if the prior smoothness parameter α is chosen large enough, then using the empirical Bayes estimator of τ allows to achieve the optimal rate of contraction. Note that [Rousseau and Szabo (2017)] find similar conditions for the case $p=\gamma=0$, stating that for $\alpha<\beta-1/2$ the rate of contraction in such model is always suboptimal.

We will investigate in the next section how this works in practice.

7. Simulation results

7.1. Volterra operator

We illustrate our results using simulated data with the Volterra operator [Halmos(1974)] following the setting of Section 2. We set $H_1 = H_2 = L^2[0,1]$. The Volterra operator, $K : L^2[0,1] \to L^2[0,1]$ is defined by

$$K\mu(x) := \int_0^x \mu(s)ds, \quad \text{and} \quad K^T\mu(x) := \int_x^1 \mu(s)ds.$$

The eigenvalues of KK^T and the orthonormal eigenbasis for the range of K are

$$k_i := [(i - \frac{1}{2})^2\pi^2]^{-\frac{1}{2}}, \quad \text{and} \quad \phi_i(x) := \sqrt{2}\sin((i - \frac{1}{2})\pi x) \quad \text{for every} \quad i \in \mathbb{N},$$

where $k_i \asymp i^{-p}$ with $p = 1$. The corresponding orthonormal eigenbasis of K^TK is

$$e_i(x) = \sqrt{2}\cos((i - \frac{1}{2})\pi x).$$

We will estimate the following approximation of $\mu_0(x)$:

$$\mu_0^N(x) := \sum_i^N \mu_{0,i}e_i(x),$$

where N is the truncation parameter and is large to ensure good approximation.

We consider a particular function with

$$\mu_{0,i} := i^{-3/2}\sin(i)$$

which belongs to S^β with $\beta = 1$ (this can be shown using Dirichlet's test, [Voxman (1981)]).

We consider the prior distribution with polynomially decaying eigenvalues and the variance that behaves polynomially, as discussed in Section 4, with

$$\lambda_i := \tau_\epsilon^2\, i^{-1-2\alpha}, \quad \text{and} \quad \sigma_i := 2i^\gamma.$$

Parameters of the prior will be specified later.

We consider several values of γ: $\gamma = -2$ (leads to parametric rate of convergence), $\gamma = -1$ (corresponds to $p+\gamma = 0$ and hence nonparametric rate of convergence), $\gamma = 0.5$ (corresponds to the rate of convergence of an ill-posed problem with $\tilde{p} = 1.5$).

Realisations of the data are simulated as follows:

$$Y_i|\mu_{0,i} \sim N(\mu_{0,i}k_i, \epsilon^2\sigma_i^2),$$

independently for $i = 1, \dots, N$ with large N.

The corresponding posterior distribution is

$$\hat{\mu}^N(x)|Y \sim N(\sum_i^N \frac{Y_ik_i\lambda_ie_i(x)}{\lambda_ik_i^2 + \epsilon^2\sigma_i^2}, \sum_i^N \frac{\epsilon^2\sigma_i^2\lambda_ie_i^2(x)}{\lambda_ik_i^2 + \epsilon^2\sigma_i^2}).$$

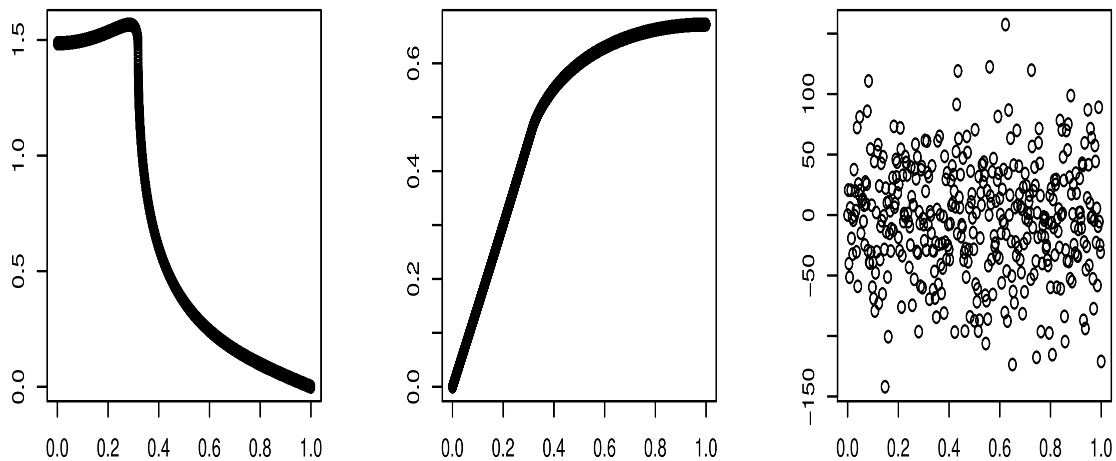

Figure 1. Graphs of $\mu_0^N(x)$, $Y_0(x)$ and $Y(x)$, (i.e. the truncated true function, the noiseless data set and a noisy data set respectively), with $\epsilon = 10^{-3/2}$ and $N = 2000$.

7.2. Non-adaptive prior

In this section we fix $\gamma = 0.5$, $\tau_\epsilon = 1$ and consider different values of α.

Figure 1 displays the (truncated) true function $\mu_0^N(x)$, along with the observed function $Y_0(x) := K\mu_0^N(x)$ and its noisy counterpart $Y(x)$.

We can therefore obtain posterior pointwise credible bands for each x. Hence, by altering ϵ and α, we can dictate the degree of noise in our model, and the smoothness of our estimator respectively.

Each of the panels in Figure 2 correspond to an independent realization of $Y(x)$, with $N = 2000$. The blue, red and green curves are the true curve ($\mu_0^N(x)$), the posterior mean and the posterior pointwise credible bands respectively. The panels also show 500 realizations from the posterior distribution for various values of x. Note, the 6 panels correspond to the following 6 values of $\alpha = (0.5, 0.75, 1, 2, 3, 5)$.

There are several conclusions that can be drawn from the panels in Figure 2. For large values of α, i.e. a prior that is too smooth, not only do the confidence bands fail to contain the true curve, they also collapse to an incorrect curve. The smaller value of α, the larger variability of the posterior. The value of α (among the considered values) that gives the posterior with the smallest uncertainty while containing the true function appears to be 0.75, which is less than $\beta = 1$ (the smoothness of $\mu_0(x)$).

Recall that as $\epsilon \to 0$, the posterior mean will converge to $\mu_0(x)$ if the prior parameters satisfy conditions of Theorem 2, and the choice of the parameters affects the rate of convergence. To that end, consider Figure 3, which is constructed in exactly the same fashion as Figure 2, albeit with $\epsilon = 10^{-4}$ instead. Nonetheless, just as in Figure

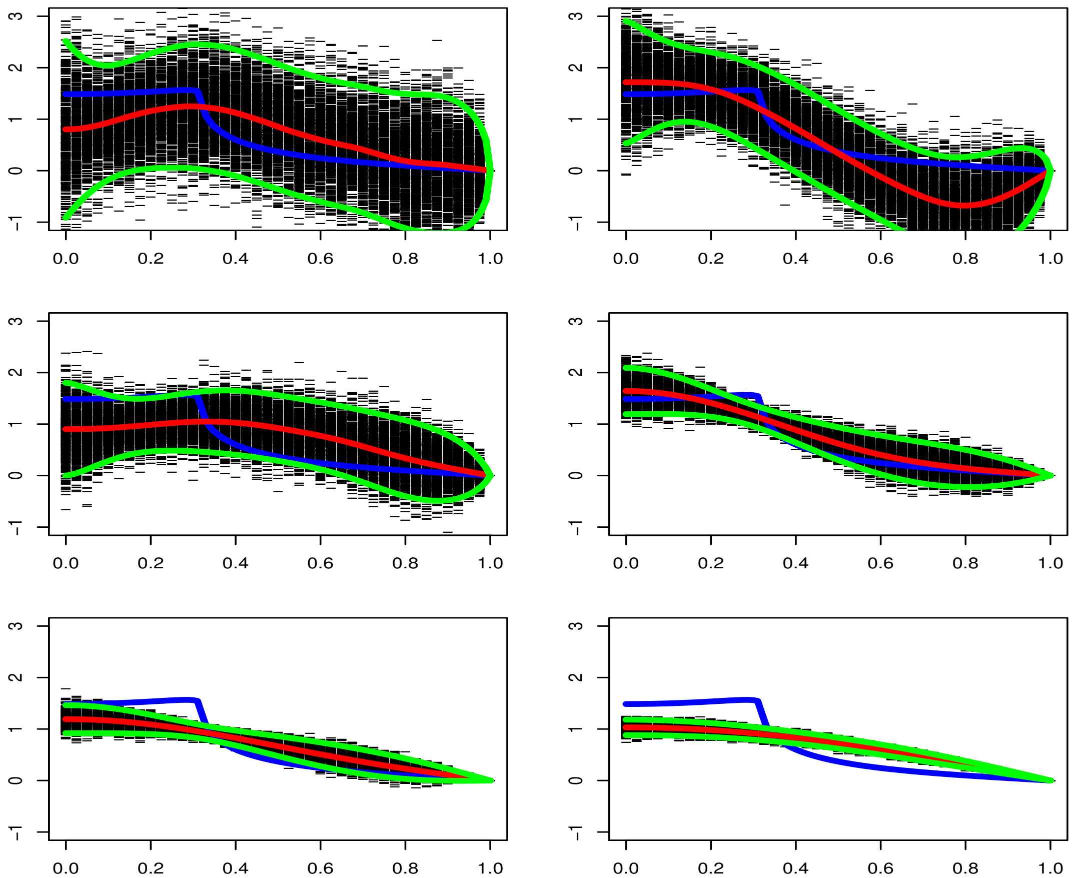

Figure 2. Plots of $\mu_0^N(x)$ (blue lines) along with the posterior mean (red line), 95% pointwise credible intervals (green curves) and 500 draws from the posterior (dashes) for $\alpha = (0.5, 0.75, 1, 2, 3, 5)$ respectively, with $\epsilon = 10^{-3/2}$ and $N = 2000$ in all cases.

2, an over-smooth prior is still inaccurate, even for a very small ϵ. However, unlike in Figure 2, the posterior mean has converged to the true function. Furthermore, the optimal alpha, ($\alpha = 0.75$), remains unchanged. In conclusion, our simulations show oversmooth priors remain inaccurate, and their posterior means continue to converge to the truth slowly, even as $\epsilon \to 0$.

7.3. Empirical Bayes posterior

Consider $0 < \beta \leq B_0$.

Case $\gamma = 0.5$. Level of ill-posedness is $\tilde{p} = p + \gamma = 1.5$. Conditions of Theorem 6 are satisfied for $\alpha \geq (B_0 - 1)/2$.

Fix $\alpha = 1$ which satisfies the above conditions for any $B_0 \leq 3$, and use the Empirical Bayes estimator of τ_ϵ defined by (14). Draws from the posterior are given in Figure 4. We can see that for $n = 10^1 0$ and $\alpha = \beta = 1$, posterior concentrates well around the true function, and that the true function is within the draws of the posterior. 95% of

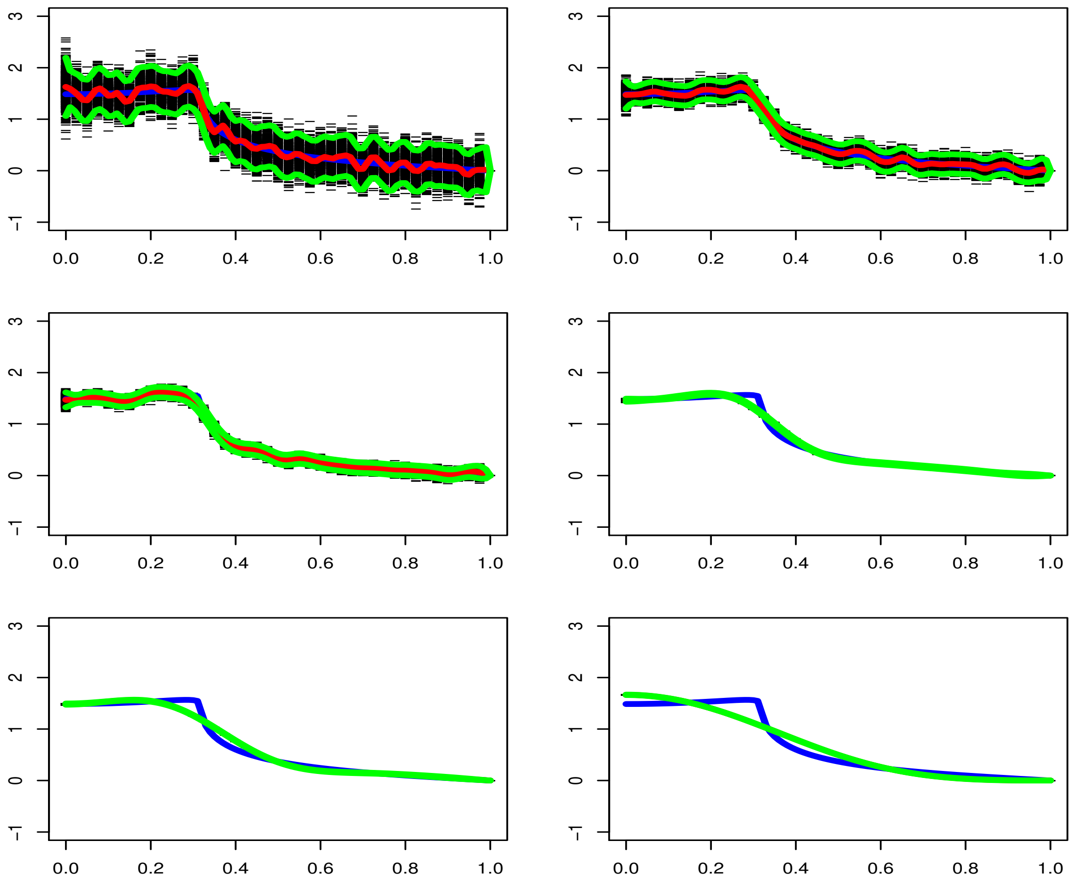

Figure 3. Plots of $\mu_0^N(x)$ (blue lines) along with the posterior mean (red line), 95% pointwise credible intervals (green curves) and 500 draws from the posterior (dashes) for $\alpha = (0.5, 0.75, 1, 2, 3, 5)$ respectively, with $\epsilon = 10^{-4}$ and $N = 2000$ in all cases.

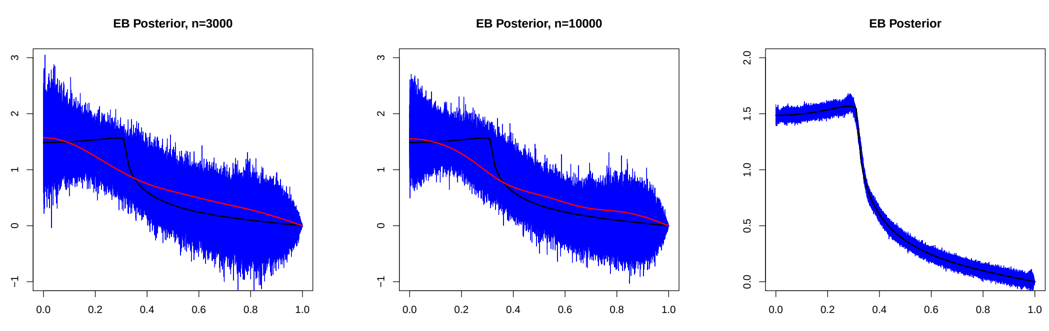

Figure 4. 100 draws from EB posterior, $n = 3000$ (left), $n = 10000$ (middle) and $n = 10^{10}$ (right), $\alpha = 1$, $\gamma = 0.5$. Black - truth, red - posterior mean.

the values $\hat{\tau}$ (over 100 simulations) are in the interval $[0.65, 0.80]$.

For larger values of α, sample size n needs to be larger for the posterior distribution to concentrate around the true function (Figure 5).

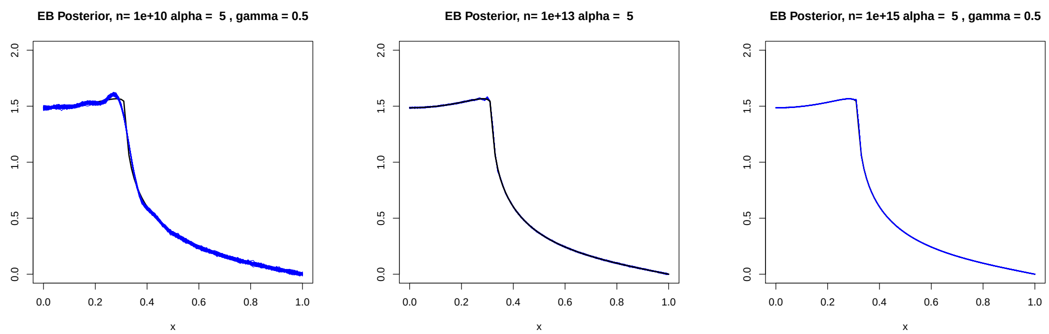

Figure 5. 100 draws from EB posterior, $\alpha = 5$ and $\gamma = 0.5$; $n = 10^{10}$ (left), $n = 10^{13}$ (middle) and $n = 10^{15}$ (right). Black - truth, red - posterior mean.

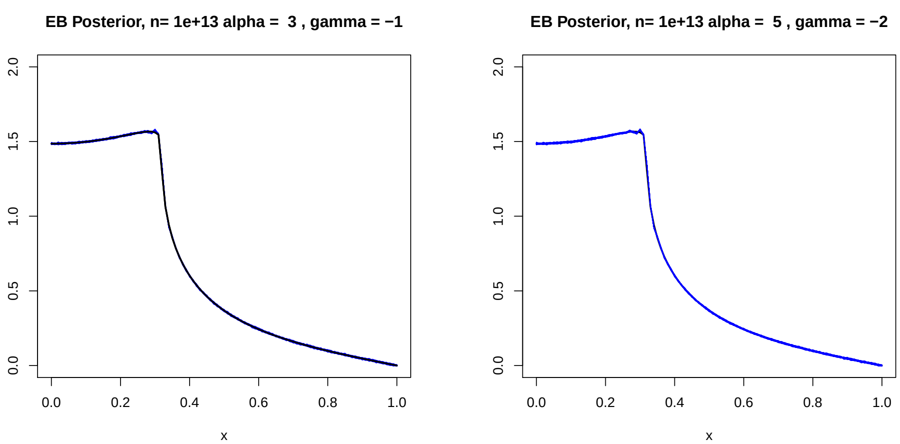

Figure 6. 100 draws from EB posterior (blue), $\alpha = 5$ and $n = 10^{13}$; $\gamma = -1$ (left) and $\gamma = -2$ (right). Black - truth.

We also consider cases $\gamma = -1$ and $\gamma = -2$. For $\gamma = -1$, the level of ill-posedness is $\tilde{p} = 0$, corresponding to the rate of convergence of direct problem. Conditions of Theorem 6 are satisfied for $\alpha \geq (B_0 - 1)/2$.

For $\gamma = -2$, the rate of contraction is parametric. Conditions of Theorem 6 are satisfied for $\alpha \geq \max((B_0 + 1)/2, 3/2)$.

Use the Empirical Bayes estimator of τ_ϵ defined by (14). Draws from the EB posterior for $\gamma = -1$ and $\gamma = -2$ with $\alpha = 5$ are given in Figure 6. In these cases, we also find that a larger sample size is needed for the posterior distribution to contract.

Boxplots of values of $\hat{\tau}$ over 100 simulations for different values of α and the considered values of γ are given in Figure 7. In each case, the sampling distribution of $\hat{\tau}$ concentrates, and the values appear to increase exponentially as functions of α. The values of $\hat{\tau}$ do not differ much for the considered different values of γ.

We also plotted values of λ_i with plugged in $\hat{\tau}$ for different values of α and γ (Figure 8). Note that the eigenvalues do not much differ for the considered values of γ, as the only effect is through $\hat{\tau}$. For each γ, the values of $\hat{\tau}$ are such that the values of

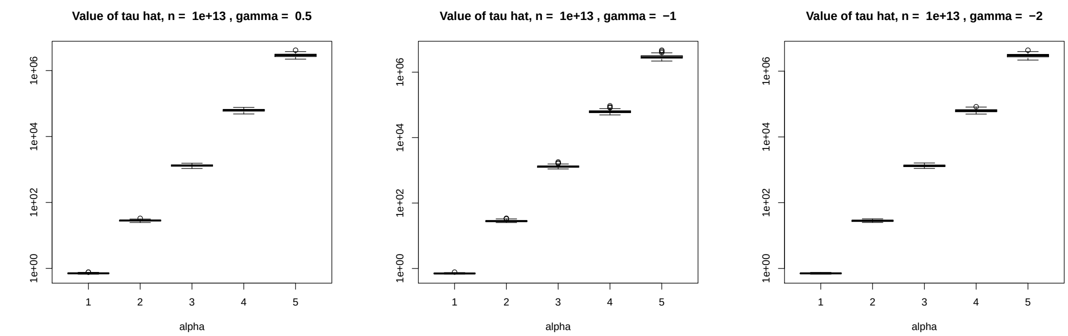

Figure 7. Boxplots of $\hat{\tau}$ over 100 draws for $n = 10^{13}$ and various values of α; $\gamma = 0.5$ (left), $\gamma = -1$ (middle) and $\gamma = -2$ (right).

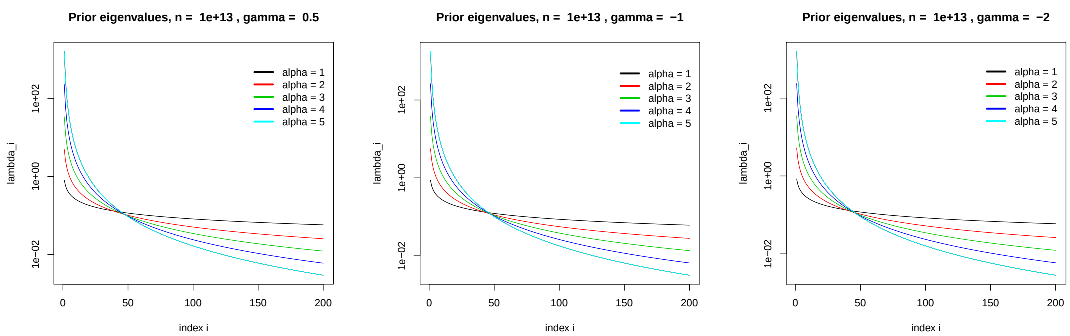

Figure 8. Values of $\sqrt{\lambda_i}$ with $\tau_\epsilon = \sqrt{\hat{\tau}}$, $n = 10^{13}$; $\gamma = 0.5$ (left), $\gamma = -1$ (middle) and $\gamma = -2$ (right) (single draw).

λ_i are the same at some index i (around $i = 50$). This is expected from Remark 3 that $\lambda_{\hat{i}_\epsilon}$ that corresponds to the optimal cutoff $\hat{i}_\epsilon$ is independent of α.

Therefore, choosing α larger than β with Empirical Bayes estimate of τ does lead to the contraction of the posterior distribution of μ but at a cost of contraction at a larger sample size compared to the case $\alpha = \beta$. This holds for various values of γ, including a case with the rate of an ill-posed inverse problem ($p + \gamma > 0$), partially regularised model ($p + \gamma = 0$) and a case where the contraction rate is parametric ($p + \gamma < -1/2$).

8. Proofs

8.1. Proof of the minimax rates

Proof: [Proof of Proposition 1] To prove Proposition 1 we need to prove the upper and the lower bound.

In order to prove the lower bound, we shall use Theorem 3 from [Belitser and Levit(1995)]. However, before stating said theorem, note the model studied in [Belitser and Levit(1995)] is defined using spectral values, i.e.

$$\tilde{Y}_i = \theta_i + \epsilon\tilde{\sigma}_i\xi, \quad \text{where} \;\; \xi \sim N(0,1),\; \tilde{\sigma}_i \geq 0 \text{ and the small parameter } \epsilon > 0. \tag{18}$$

However, (18) is equivalent to the heterogeneous model (1) if

$$\tilde{Y}_i := \frac{Y_i}{k_i}, \quad \theta_i := \mu_i \quad \text{and} \quad \tilde{\sigma}_i := \frac{\sigma_i}{k_i} \asymp i^{\tilde{p}}$$

where $\tilde{p} = p + \gamma$.

Furthermore, in [Belitser and Levit(1995)], it is assumed the true parameter $\theta_0 \in \Theta$, where

$$\Theta = \Theta(Q) = \{\theta : \sum_{i=1}^{\infty} a_i^2 \, \theta_i^2 \leq Q\}$$

and (a_i) is a non-negative sequence converging to infinity. As we assume $\mu_0 \in S^\beta(A)$, where $\beta > 0$, we take $a_i = i^\beta$ which implies $\Theta = S^\beta(A)$ with $Q = A^2$.

Our goal is to find the minimax rate of convergence in L^2 norm:

$$\varepsilon^{*2} = r_\epsilon = r_\epsilon(\Theta) = \inf_{\hat{\theta}} \sup_{\theta_0 \in \Theta} \mathbb{E}_{\theta_0} ||\hat{\theta} - \theta_0||_2^2.$$

We can find this by using [Belitser and Levit(1995)]'s *Theorem 3*, which is as follows (in our notation).

Theorem 7 *Define c_ϵ to be the solution of the equation $\epsilon^2 \sum_{i=1}^{\infty} \tilde{\sigma}_i^2 a_i (1 - c a_i)_+ = cQ$ and $N := N_\epsilon(\Theta) = \max\{i : a_i \leq c_\epsilon^{-1}\}$. If condition*

$$\log \epsilon^{-1} \frac{\sum_{i=1}^{\infty} a_i^2 \tilde{\sigma}_i^4 (1 - c_\epsilon a_i)_+^2}{(\sum_{i=1}^{\infty} a_i \tilde{\sigma}_i^2 (1 - c_\epsilon a_i)_+)^2} = o(1), \quad \epsilon \to 0, \tag{19}$$

holds, then

$$r_\epsilon = \epsilon^2 \sum_{i=1}^{N} \tilde{\sigma}_i^2 - \epsilon^2 (c_\epsilon \sum_{i=1}^{N} \tilde{\sigma}_i^2 a_i)(1 + o(1)), \quad \epsilon \to 0.$$

In order to use this theorem, we must first find c_ϵ and N. Note

$$(1 - c_\epsilon a_i)_+ = (1 - c_\epsilon i^\beta)_+ = 0 \iff i \geq c_\epsilon^{-\frac{1}{\beta}} \implies N = \lfloor c_\epsilon^{-\frac{1}{\beta}} \rfloor \tag{20}$$

Consequently,

$$\epsilon^2 \sum_{i=1}^{\infty} \tilde{\sigma}_i^2 a_i (1 - c_\epsilon a_i)_+ = \epsilon^2 \sum_{i=1}^{N} i^{2\tilde{p}+\beta} (1 - c_\epsilon i^\beta) = \epsilon^2 \sum_{i=1}^{N} i^{2\tilde{p}+\beta} - c_\epsilon \epsilon^2 \sum_{i=1}^{N} i^{2\tilde{p}+2\beta}$$

Note that $N \to \infty$ and $c_\epsilon \to 0$ as $\epsilon \to 0$. Suppose this is not true, i.e. $c_\epsilon \geq c_1 > 0$. Then, N is finite, and hence the equation defining c_ϵ implies

$$\epsilon^2 \left[\sum_{i=1}^{N} i^{2\tilde{p}+\beta} (1 - c_\epsilon i^\beta) \right] = \epsilon^2 C(c_\epsilon, \tilde{p}, \beta) = c_\epsilon Q$$

which contradicts the assumption that $\epsilon \to 0$, as $C(c_\epsilon, \tilde{p}, \beta)$, c_ϵ, Q are positive finite constants. Hence, $N \to \infty$.

Note, we can use the following equation (derived by bounding a sum by its integral) to bound the above sums,

$$\sum_{i=1}^{N} i^{\kappa} = \frac{N^{\kappa+1}}{\kappa+1}(1+o(1)) \quad \text{as} \quad N \to \infty, \quad \text{if} \quad \kappa > -1. \tag{21}$$

Consider the case

$$2\tilde{p} + \beta > -1 \Longrightarrow p + \gamma > -\frac{1+\beta}{2} \tag{22}$$

as assumed in the proposition. Consequently, one can show via (21) that

$$\begin{aligned}\epsilon^2 \sum_{i=1}^{\infty} \tilde{\sigma}_i^2 a_i (1 - c_\epsilon a_i)_+ &= \epsilon^2 \frac{N^{2\tilde{p}+\beta+1}}{2\tilde{p}+\beta+1}(1+o(1)) - c_\epsilon \epsilon^2 \frac{N^{2\tilde{p}+2\beta+1}}{2\tilde{p}+2\beta+1}(1+o(1)) \\ &= \epsilon^2 c_\epsilon^{-\frac{c_1}{\beta}} \frac{\beta}{c_1 c_2}(1+o(1))\end{aligned}$$

using (20), where $c_1 := 2\tilde{p}+\beta+1$ and $c_2 := 2\tilde{p}+2\beta+1$.

Consequently,

$$\epsilon^2 \sum_{i=1}^{\infty} \tilde{\sigma}_i^2 a_i (1 - c_\epsilon a_i)_+ = c_\epsilon Q \iff \epsilon^2 c_\epsilon^{-\frac{c_1}{\beta}} \frac{\beta}{c_1 c_2} = c_\epsilon Q \Longrightarrow$$

$$c_\epsilon = \left(\frac{\epsilon^2 \beta}{Q c_1 c_2}\right)^{\frac{\beta}{c_2}}(1+o(1)) \quad \text{and} \quad N = \lfloor c_\epsilon^{-\frac{1}{\beta}} \rfloor = \left(\frac{\epsilon^2 \beta}{Q c_1 c_2}\right)^{-\frac{1}{c_2}}(1+o(1)). \tag{23}$$

Now we need to verify condition (19):

$$\frac{\sum_{i=1}^{\infty} a_i^2 \tilde{\sigma}_i^4 (1 - c_\epsilon a_i)_+^2}{(\sum_{i=1}^{\infty} a_i \tilde{\sigma}_i^2 (1 - c_\epsilon a_i)_+)^2} \le \frac{\sum_{i=1}^{N} a_i^2 \tilde{\sigma}_i^4}{(c_\epsilon^{-\frac{c_1}{\beta}} \frac{\beta}{c_1 c_2}(1+o(1)))^2} = \mathcal{O}\left(\frac{N^{4\tilde{p}+2\beta+1}}{N^{4\tilde{p}+2\beta+2}}\right) = \mathcal{O}((\epsilon^2)^{\frac{1}{c_2}})$$

which tends to 0 as $\epsilon \to 0$ since $c_2 = 2\tilde{p}+2\beta+1 > 0$.

Therefore, we can now use the theorem to find the rates for the three different cases in Proposition 1. Thus,

$$r_\epsilon \asymp \epsilon^2 \sum_{i=1}^{N} i^{2\tilde{p}} - \epsilon^2 (c_\epsilon \sum_{i=1}^{N} i^{2\tilde{p}+\beta})$$

Note that we need $\epsilon^* = \sqrt{r_\epsilon}$. Since $2\tilde{p}+\beta > -1$, the latter sum is $c_\epsilon \frac{N^{2\tilde{p}+\beta+1}}{2\tilde{p}+\beta+1}](1+o(1))$.

Subsequently, in order to bound the above terms, we consider 3 cases.

<u>Case 1</u>: $2\tilde{p} > -1 \Longrightarrow 2\tilde{p}+\beta > -1$. Hence, we can use (21) and (23) to show

$$r_\epsilon \asymp \epsilon^2 \left[\frac{N^{2\tilde{p}+1}}{2\tilde{p}+1} - c_\epsilon \frac{N^{2\tilde{p}+\beta+1}}{2\tilde{p}+\beta+1}\right] = \epsilon^2 c_\epsilon^{-\frac{2\tilde{p}+1}{\beta}} \frac{\beta}{(2\tilde{p}+1)c_1} = \mathcal{O}((\epsilon^2)^{\frac{2\beta}{c_2}}).$$

<u>Case 2</u>: $2\tilde{p} = -1 \Longrightarrow 2\tilde{p}+\beta > -1$. Bounding a sum via its integral, we can obtain the following

$$\sum_{i=1}^{N} i^{-1} = \log N + C_e + o(1), \quad \text{as} \quad N \to \infty, \quad \text{where} \ \ C_e \text{ is the Euler constant.}$$

Consequently, since $N \to \infty$ as $\epsilon \to 0$, we can use the above and (23) to derive

$$r_\epsilon \asymp \epsilon^2[\log N - c_\epsilon \frac{N^\beta}{\beta}] = \mathcal{O}(\epsilon^2 \log N) = \mathcal{O}(\epsilon^2 \log|\epsilon|).$$

Case 3: $2\tilde{p} < -1$. Using the following asymptotic expansion,

$$\sum_{i=1}^{N} i^{-\kappa-1} = \frac{1}{\kappa}(1+o(1)) \quad \text{as} \quad N \to \infty, \quad \text{if} \quad \kappa > 0,$$

and (21), we have

$$r_\epsilon \asymp \epsilon^2[\frac{1}{-2\tilde{p}-1} - \frac{c_\epsilon N^{2\tilde{p}+\beta+1}}{2\tilde{p}+\beta+1}] = \mathcal{O}(\epsilon^2[1-(\epsilon^2)^{-\frac{2\tilde{p}+1}{c_2}}]) = \mathcal{O}(\epsilon^2)$$

as $\epsilon \to 0$, since $2\tilde{p} < -1 \implies \frac{2\beta}{c_2} > 1$.

To avoid the constraint on β for $p+\gamma+1/2 < 0$, we apply Theorems 2.1 and 2.2 from [Tsybakov (2008)] with $d(\mu_2, \mu_1) = ||\mu_2 - \mu_1||_2$ and $\Theta = S^\beta(A)$. For that we need to find two elements μ_2, μ_1 in $S^\beta(A)$ such that $d(\mu_2, \mu_1) \geq B\psi_n$ for some $B > 0$ and $KL(P_{\mu_1}, P_{\mu_2}) \leq \alpha < \infty$ where ψ_n is the rate defined in the proposition. Here P_μ is the probability distribution of Y generated by model (1). Then, for any estimator $\hat{\mu}$ and $\mu_0 \in S^\beta(A)$,

$$E_{\mu_0}[\psi_n^{-2}||\hat{\mu} - \mu_0||_2^2] \geq (B/2)^2 \max\left(\frac{1}{4}\exp(-\alpha), \frac{1-\sqrt{\alpha/2}}{2}\right).$$

We take $\mu_1 = 0$ and $\mu_2 = \sum_i \mu_{2,i} e_i$ such that $\mu_{2,i} = B\psi_n$ for $i = i_0 \geq 1$ and $\mu_{2,i} = 0$ otherwise. Such μ_2 belongs to $S^\beta(A)$ if $\sum_i i^{2\beta}\mu_{2,i}^2 = i_0^{2\beta}(B\psi_n)^2 \leq A^2$, i.e. if $i_0 \leq (A/B)^{1/\beta}\psi_n^{-1/\beta}$. Also, $d(\mu_2, \mu_1) = \sqrt{\sum_i \mu_{2,i}^2} = B\psi_n$. Using spectral decomposition, it is easy to compute the Kullback-Leibler distance:

$$KL(P_{\mu_1}, P_{\mu_2}) = \frac{1}{2}\left[\sum_i \frac{k_i^2 \mu_{2,i}^2}{\epsilon^2\sigma_i^2} - 1\right] \leq \frac{1}{2}\left[c_k^2 B^{-2} i_0^{-2(p+\gamma)}\psi_n^2\epsilon^{-2} - 1\right].$$

As $p+\gamma+1/2 < 0$, take $\psi_n = \epsilon$ and i_0 such that $i_0 < c_k^{2/(p+\gamma)}B^{-2/(p+\gamma)}$, then $\alpha = 0.5\left[c_k^2 B^{-2} i_0^{-2(p+\gamma)} - 1\right]$, so that the rate is $\psi_n = \epsilon$.

For the upper bound, consider the estimator $\hat{\mu} = y_i/k_i I(i \leq i_1)$ with i_1 to be specified later. Then,

$$\begin{aligned} E_{\mu_0}||\hat{\mu} - \mu_0||_2^2 &= E_{\mu_0}\sum_{i\leq i_1}(y_i/k_i - \mu_{0,1})^2 + \sum_{i>i_1}\mu_{0,1}^2 \leq \sum_{i\leq i_1}\epsilon^2\sigma_i^2/k_i^2 + i_1^{-2\beta}\sum_{i>i_1} i^{2\beta}\mu_{0,1}^2 \\ &\leq C\epsilon^2\sum_{i\leq i_1} i^{2(p+\gamma)} + i_1^{-2\beta}A^2. \end{aligned}$$

If $p+\gamma+1/2 > 0$, then $\sum_{i\leq i_1} i^{2(p+\gamma)} \leq C i_1^{2(p+\gamma+1/2)}$, and

$$E_{\mu_0}||\hat{\mu} - \mu_0||_2^2 \leq C\epsilon^2 i_1^{2(p+\gamma+1/2)} + i_1^{-2\beta}A^2 \leq C[\epsilon^{-2}]^{-2\beta/(2p+2\gamma+2\beta+1)}$$

with $i_1 = \epsilon^{-1/(p+\gamma+1/2+\beta)}$. If $p+\gamma = 0$, this is the minimax rate for the direct problem.

If $p+\gamma+1/2=0$, then $\sum_{i\le i_1} i^{2(p+\gamma)} \le C\log i_1$, and

$$E_{\mu_0}||\hat{\mu}-\mu_0||_2^2 \le C\epsilon^2 \log i_1 + i_1^{-2\beta}A^2 \le C\epsilon^2\log|\epsilon|$$

with $i_1 = \epsilon^{-1/\beta}$.

If $p+\gamma+1/2<0$, then $\sum_{i\le i_1} i^{2(p+\gamma)} \le C$, and

$$E_{\mu_0}||\hat{\mu}-\mu_0||_2^2 \le C\epsilon^2 + i_1^{-2\beta}A^2 \le C\epsilon^2$$

with $i_1 = \epsilon^{-1/\beta}$.

□

8.2. Proof of the main theorem on contraction of posterior distribution

Proof: [Proof of Theorem 1.]

The main tool we are going to use is Markov inequality:

$$\mathbb{P}(||\mu-\mu_0||_2 \ge M\varepsilon \mid Y) \le M^{-2}\varepsilon^{-2}\mathbb{E}(||\mu-\mu_0||_2^2 \mid Y).$$

Note that for a random variable $\xi \ge 0$, $\mathbb{P}(\xi\to 0)=1$ follows from $\mathbb{E}\xi \to 0$. Hence, if we show that $\mathbb{E}_{\mu_0}\mathbb{E}(||\mu-\mu_0||_2^2 \mid Y) \le C\varepsilon^2$ for some $C>0$ independent of ε then ε is the rate of contraction of the posterior distribution.

Under Assumption 1, using Parseval's identity, we have that

$$\mathbb{E}(||\mu-\mu_0||^2 \mid Y) = \sum_i \mathbb{E}[(\mu_i-\mu_{0i})^2 \mid Y] = \sum_i [Var(\mu_i \mid Y) + (\mathbb{E}[\mu_i| \mid Y]-\mu_{0i})^2]$$

Taking the expected value with respect to the true distribution of the data and using the explicit form of the posterior distribution (5), we have

$$\begin{aligned}
\mathbb{E}_{\mu_0}[Var(\mu_i \mid Y)] + \mathbb{E}_{\mu_0}[(\mathbb{E}[\mu_i| \mid Y]-\mu_{0i})^2] &= \mathbb{E}_{\mu_0}\left[\frac{Y_i k_i \lambda_i}{\lambda_i k_i^2+\epsilon^2\sigma_i^2} - \mu_{0i}\right]^2 + \frac{\sigma_i^2\lambda_i}{\epsilon^{-2}\lambda_i k_i^2+\sigma_i^2} \\
&= \frac{\epsilon^2\sigma_i^2 k_i^2\lambda_i^2}{[\lambda_i k_i^2+\epsilon^2\sigma_i^2]^2} + \mu_{0i}^2\left[\frac{k_i^2\lambda_i}{\lambda_i k_i^2+\epsilon^2\sigma_i^2}-1\right]^2 + \frac{\sigma_i^2\lambda_i}{\epsilon^{-2}\lambda_i k_i^2+\sigma_i^2} \\
&\asymp \frac{\sigma_i^2\lambda_i}{\epsilon^{-2}\lambda_i k_i^2+\sigma_i^2} + \mu_{0i}^2\left[\frac{k_i^2\lambda_i}{\lambda_i k_i^2+\epsilon^2\sigma_i^2}-1\right]^2
\end{aligned}$$

as the first term is less than the third one.

Recall that $\sigma_i^2/[\lambda_i k_i^2]$ is an increasing sequence. Denote $i_\epsilon = \max\{i : \sigma_i^2/[\lambda_i k_i^2] \le \epsilon^{-2}\}$. Then,

$$S_1 = \sum_i \frac{\sigma_i^2\lambda_i}{\epsilon^{-2}\lambda_i k_i^2+\sigma_i^2} \asymp \epsilon^2\sum_{i\le i_\epsilon}\sigma_i^2 k_i^{-2} + \sum_{i>i_\epsilon}\lambda_i,$$

and

$$\begin{aligned}
S_2 &= \sum_i \mu_{0i}^2\left[\frac{\epsilon^2\sigma_i^2}{k_i^2\lambda_i+\epsilon^2\sigma_i^2}\right]^2 = ||\mu_0||_{S^\beta}^2\sum_i \bar{\mu}_{0,i}^2\left[\frac{\epsilon^2\sigma_i^2 i^{-\beta}}{k_i^2\lambda_i+\epsilon^2\sigma_i^2}\right]^2 \\
&\asymp ||\mu_0||_{S^\beta}^2\epsilon^4\sum_{i\le i_\epsilon}\bar{\mu}_{0,i}^2\left[\frac{\sigma_i^2 i^{-\beta}}{k_i^2\lambda_i}\right]^2 + ||\mu_0||_{S^\beta}^2\sum_{i>i_\epsilon}\bar{\mu}_{0,i}^2 i^{-2\beta}
\end{aligned}$$

$$\leq ||\mu_0||^2_{S^\beta}\left[\epsilon^4 \max_{i\leq i_\epsilon}\left[\frac{\sigma_i^2 i^{-\beta}}{k_i^2\lambda_i}\right]^2 + Ci_\epsilon^{-2\beta}\right]$$

where $\bar{\mu}^2_{0,i} := \mu^2_{0,i} i^{2\beta}/||\mu_0||^2_{S^\beta}$ and $||\mu_0||^2_{S^\beta} = \sum_i \mu^2_{0i} i^{2\beta}$. The lower bound can be proved by taking μ_0 such that $\bar{\mu}_{0i} = 1$ for one of the $i \leq i_\epsilon$ and $\bar{\mu}_{0i} = 0$ otherwise to get the first term, and $\bar{\mu}_{0i} = 0$ for $i \neq i_\epsilon + 1$ and $\bar{\mu}_{0i} = 1$ for $i = i_\epsilon + 1$ to get the second term (up to a constant).

Hence,

$$S_2 \asymp ||\mu_0||^2_{S^\beta}\left[\epsilon^4 \max_{i\leq i_\epsilon}\left[\frac{\sigma_i^2 i^{-\beta}}{k_i^2\lambda_i}\right]^2 + i_\epsilon^{-2\beta}\right].$$

Combining these results together, we obtain

$$\mathbb{E}_{\mu_0}\mathbb{E}(||\mu-\mu_0||^2 \mid Y) \asymp \epsilon^2\sum_{i\leq i_\epsilon}\sigma_i^2 k_i^{-2} + \sum_{i>i_\epsilon}\lambda_i + \epsilon^4 \max_{i\leq i_\epsilon}\left[\frac{\sigma_i^2 i^{-\beta}}{k_i^2\lambda_i}\right]^2 + i_\epsilon^{-2\beta}$$

Hence, setting ε such that

$$\varepsilon^{-2}\mathbb{E}_{\mu_0}\mathbb{E}(||\mu-\mu_0||^2 \mid Y) \asymp \varepsilon^{-2}\left[\epsilon^2\sum_{i\leq i_\epsilon}\sigma_i^2 k_i^{-2} + \sum_{i>i_\epsilon}\lambda_i + \epsilon^4 \max_{i\leq i_\epsilon}\left[\frac{\sigma_i^2 i^{-\beta}}{k_i^2\lambda_i}\right]^2 + i_\epsilon^{-2\beta}\right] = O(1)$$

as $\epsilon \to 0$ ensures that

$$\mathbb{E}_{\mu_0}\mathbb{P}(\{\mu : ||\mu-\mu_0|| \geq M\varepsilon|Y\}) \leq M^{-2}\varepsilon^{-2}\mathbb{E}_{\mu_0}\mathbb{E}(||\mu-\mu_0||^2 \mid Y) \to 0 \quad \text{as } \epsilon \to 0$$

for every $M \to \infty$.

□

8.3. *Plug-in estimator*

Proof: [Proof of Theorem 3]

The proof is following the lines of the proof of Theorem 1.

Define $\Omega_\sigma = \{|\hat{\sigma}_i^2 - \sigma_i^2| \leq c_0\epsilon_\sigma,\ i = 1, 2, \ldots\}$. Under assumption (6), $P(\Omega_\sigma) \to 1$ as $\epsilon_\sigma \to 0$. Then, on Ω_σ, $\hat{\sigma}_i^2 \leq \sigma_i^2 + c_0\epsilon_\sigma$ and

$$\tilde{\sigma}_i^2 = \max(\hat{\sigma}_i^2, a_\sigma) \geq \max(\sigma_i^2 - c_0\epsilon_\sigma, a_\sigma).$$

Consider the case $a_\sigma = c_0\epsilon_\sigma$.

Then,

$$\mathbb{E}_{\mu_0,V}\mathbb{P}(||\mu-\mu_0||_2 \geq M\varepsilon \mid Y, \hat{V}) \leq \mathbb{E}_{\mu_0,V}\mathbb{P}(||\mu-\mu_0||_2 \geq M\varepsilon \mid Y, \hat{V})I(\Omega_\sigma) + P_{\mu_0,V}(\Omega_\sigma)$$

and the latter term goes to 0 by assumption (6). For the former term, Markov's inequality implies:

$$\mathbb{P}(||\mu-\mu_0||_2 \geq M\varepsilon \mid Y, \hat{V}) \leq M^{-2}\varepsilon^{-2}\mathbb{E}(||\mu-\mu_0||_2^2 \mid Y, \hat{V}),$$

and under Assumption 1, using Parseval's identity, we have that

$$\mathbb{E}(||\mu-\mu_0||^2 \mid Y,\hat{V}) = \sum_i \mathbb{E}[(\mu_i-\mu_{0i})^2 \mid Y,\hat{V}] = \sum_i [Var(\mu_i \mid Y,\hat{V}) + (\mathbb{E}[\mu_i \mid Y,\hat{V}]-\mu_{0i})^2].$$

Taking expected value with respect to the true distribution of Y (under the assumption that it is independent of $(\hat{\sigma}_i)$) and using the explicit form of the posterior distribution (5), we have on Ω_σ

$$\begin{aligned}
&\mathbb{E}_{\mu_0}[Var(\mu_i \mid Y,\tilde{V})] + \mathbb{E}_{\mu_0}[(\mathbb{E}[\mu_i| \mid Y,\tilde{V}]-\mu_{0i})^2] \\
&= \frac{\epsilon^2\sigma_i^2 k_i^2\lambda_i^2}{[\lambda_i k_i^2+\epsilon^2\tilde{\sigma}_i^2]^2} + \mu_{0i}^2\left[\frac{\epsilon^2\tilde{\sigma}_i^2}{\lambda_i k_i^2+\epsilon^2\tilde{\sigma}_i^2}\right]^2 + \frac{\tilde{\sigma}_i^2\lambda_i}{\epsilon^{-2}\lambda_i k_i^2+\tilde{\sigma}_i^2} \\
&\le \frac{\epsilon^2\lambda_i^2 k_i^2\sigma_i^2}{[\lambda_i k_i^2+\epsilon^2\max(\sigma_i^2-c_0\epsilon_\sigma, c_0\epsilon_\sigma)]^2} + \frac{[\sigma_i^2+c_0\epsilon_\sigma]\lambda_i}{\epsilon^{-2}\lambda_i k_i^2+[\sigma_i^2+c_0\epsilon_\sigma]} \\
&+\mu_{0i}^2\left[\frac{[\sigma_i^2+c_0\epsilon_\sigma]}{\lambda_i k_i^2/\epsilon^2+[\sigma_i^2+c_0\epsilon_\sigma]}\right]^2.
\end{aligned}$$

Denote $I_\epsilon(a)=\{i:\ \sigma_i^2/[\lambda_i k_i^2] > a\epsilon^{-2}\}$ and $I_\sigma(a)=\{i:\ \sigma_i^2 < ac_0\epsilon_\sigma\}$.

Define also $i_\epsilon(a)=\min\{i:\ \sigma_i^2/[\lambda_i k_i^2] > a\epsilon^{-2}\}$. Note that if $\sigma_i^2/[\lambda_i k_i^2]$ increases then $I_\epsilon(a)=\{i\ge i_\epsilon(a)\}$. And if σ_i decreases then we can write $I_\sigma(a)=\{i\ge i_\sigma(a)\}$ where $i_\sigma(a)=\max\{i:\ \sigma_i^2\le ac_0\epsilon_\sigma\}$. Note that $I_\epsilon(a_1)\subseteq I_\epsilon(a_2)$ and $I_\sigma(a_1)\subseteq I_\sigma(a_2)$ if $a_1>a_2$. Then,

$$\begin{aligned}
S_1 &= \sum_i \frac{\epsilon^2\lambda_i^2 k_i^2\sigma_i^2}{[\lambda_i k_i^2+\epsilon^2\max(\sigma_i^2-c_0\epsilon_\sigma, c_0\epsilon_\sigma)]^2} \\
&\le \epsilon^2\sum_{i\in\bar{I}_\epsilon(1)}\sigma_i^2/k_i^2 + \epsilon_\sigma^{-2}\epsilon^{-2}\sum_{i\in I_\epsilon(1)\cap I_\sigma(2)}\lambda_i^2 k_i^2\sigma_i^2 + \sum_{i\in I_\epsilon(1)\cap\bar{I}_\sigma(2)}\frac{\epsilon^2\lambda_i^2 k_i^2\sigma_i^2}{[\lambda_i k_i^2+\epsilon^2\sigma_i^2/2]^2}.
\end{aligned}$$

The first and the last terms are the same as in the case of no plug-in, up to a constant in the indices. In the polynomial case, the second term is

$$\begin{aligned}
&\epsilon_\sigma^{-2}\tau_\epsilon^4\epsilon^{-2}\sum_{i>\max(i_\epsilon(1),i_\sigma)} i^{-4\alpha-2-2p+2\gamma} \le \epsilon_\sigma^{-2}\tau_\epsilon^4\epsilon^{-2}[\max(i_\epsilon(1),i_\sigma)]^{-4\alpha-1-2p+2\gamma} \\
&= \epsilon_\sigma^{-2}\tau_\epsilon^4\epsilon^{-2}[\max[(\tau_\epsilon^2\epsilon^{-2})^{1/(2\tilde{p}+2\alpha+1)}, \epsilon_\sigma^{1/(2\gamma)}]]^{-4\alpha-1-2p+2\gamma}
\end{aligned}$$

If $(\tau_\epsilon^2\epsilon^{-2})^{1/(2\tilde{p}+2\alpha)} > \epsilon_\sigma^{1/(2\gamma)}$ then the upper bound is

$$\epsilon_\sigma^{-2}\tau_\epsilon^4\epsilon^{-2}\big(\tau_\epsilon^2\epsilon^{-2}\big)^{-(4\alpha+1+2p-2\gamma)/(2\tilde{p}+2\alpha+1)} = \epsilon_\sigma^{-2}\tau_\epsilon^2\big(\tau_\epsilon^2\epsilon^{-2}\big)^{-(2\alpha-2\gamma)/(2\tilde{p}+2\alpha+1)} = \epsilon_\sigma^{-2}\epsilon^2\big(\tau_\epsilon^2\epsilon^{-2}\big)^{(1+2p)/(2\tilde{p}+2\alpha+1)}$$

which tends to 0 if $\epsilon^{2[(2\alpha+6\gamma)]} < \tau_\epsilon^{-2(1+2p-4\gamma)}$.

If $(\tau_\epsilon^2\epsilon^{-2})^{1/(2\tilde{p}+2\alpha)} < \epsilon_\sigma^{1/(2\gamma)}$ then the upper bound is

$$\epsilon_\sigma^{-2}\tau_\epsilon^4\epsilon^{-2}\epsilon_\sigma^{(-4\alpha-1-2p+2\gamma)/(2\gamma)} = \tau_\epsilon^4\epsilon^{-2}\epsilon_\sigma^{(-4\alpha-1-2p-2\gamma)/(2\gamma)}$$

which tends to 0 if

$$\tau_\epsilon^4\epsilon^{-2}\epsilon_\sigma^{(-4\alpha-1-2p-2\gamma)/(2\gamma)} = o(1)$$

It holds if $(\tau_\epsilon^2\epsilon^{-2})^{1/(2\tilde{p}+2\alpha)} < (\tau_\epsilon^4\epsilon^{-2})^{1/(4\alpha+1+2\tilde{p})} = o(1)\epsilon_\sigma^{1/(2\gamma)}$

The next term is

$$\begin{aligned}
S_2 &\leq \sum_{i \in I_\sigma(2)} \frac{3\epsilon^2 \lambda_i c_0 \epsilon_\sigma}{\lambda_i k_i^2 + 3\epsilon^2 c_0 \epsilon_\sigma} + \sum_{i \in \bar{I}_\sigma(2)} \frac{1.5\epsilon^2 \lambda_i \sigma_i^2}{\lambda_i k_i^2 + 1.5\epsilon^2 \sigma_i^2} \\
&\leq \sum_{i \in I_\sigma(2)\,\&\, i \in I_{\sigma\epsilon}(1/3)} \lambda_i + 3\epsilon^2 c_0 \epsilon_\sigma \sum_{i \in I_\sigma(2)\,\&\, i \in \bar{I}_{\sigma\epsilon}(1/3)} k_i^{-2} + 1.5\epsilon^2 \sum_{i \in \bar{I}_\sigma(2)\,\&\, i \in \bar{I}_\epsilon(2/3)} \sigma_i^2 k_i^{-2} \\
&\quad + \sum_{i \in \bar{I}_\sigma(2)\,\&\, i \in I_\epsilon(2/3)} \lambda_i
\end{aligned}$$

where $I_{\sigma\epsilon}(a) = \{i : \; c_0\epsilon_\sigma/[\lambda_i k_i^2] > a\epsilon^{-2}\}$.

Combining these terms, we obtain

$$\begin{aligned}
S_1 + S_2 &\leq 2\epsilon^2 \sum_{i \in \bar{I}_\epsilon(2) \cup I_\sigma(2)} \sigma_i^2 k_i^{-2} + 3\epsilon^2 c_0 \epsilon_\sigma \sum_{i \in I_\sigma(2)\,\&\, i \in \bar{I}_{\sigma\epsilon}(1/3)} k_i^{-2} \\
&\quad + \sum_{i \in I_\epsilon(2/3)} \lambda_i + \sum_{i \in I_\sigma(2)\,\&\, i \in I_{\sigma\epsilon}(1/3)} \lambda_i
\end{aligned}$$

The remaining term is

$$\begin{aligned}
S_3 &= ||\mu_0||^2_{S^\beta} \sum_i \bar{\mu}_{0,i}^2 \left[\frac{\epsilon^2[\sigma_i^2 + c_0\epsilon_\sigma] i^{-\beta}}{k_i^2 \lambda_i + \epsilon^2[\sigma_i^2 + c_0\epsilon_\sigma]} \right]^2 \\
&\leq ||\mu_0||^2_{S^\beta} \epsilon^4 \sum_{i \in \bar{I}_\epsilon(1) \cup \bar{I}_{\sigma\epsilon}(1)} \bar{\mu}_{0,i}^2 \left[\frac{[\sigma_i^2 + c_0\epsilon_\sigma] i^{-\beta}}{k_i^2 \lambda_i} \right]^2 + ||\mu_0||^2_{S^\beta} \sum_{i \in I_\epsilon(1) \cap I_{\sigma\epsilon}(1)} \bar{\mu}_{0,i}^2 i^{-2\beta} \\
&\leq ||\mu_0||^2_{S^\beta} \left[\epsilon^4 \max_{i \in \bar{I}_\epsilon(1) \cup \bar{I}_{\sigma\epsilon}(1)} \left[\frac{[\sigma_i^2 + c_0\epsilon_\sigma] i^{-\beta}}{k_i^2 \lambda_i} \right]^2 + C \max_{i \in I_\epsilon(1) \cap I_{\sigma\epsilon}(1)} i^{-2\beta} \right]
\end{aligned}$$

where $\bar{\mu}_{0,i}^2 := \mu_{0,i}^2 i^{2\beta}/||\mu_0||^2_{S^\beta}$ and $||\mu_0||^2_{S^\beta} = \sum_i \mu_{0i}^2 i^{2\beta}$.

We can rewrite the first term as follows:

$$\max_{i \in \bar{I}_\epsilon(1) \cup \bar{I}_{\sigma\epsilon}(1)} \left[\frac{[\sigma_i^2 + c_0\epsilon_\sigma] i^{-\beta}}{k_i^2 \lambda_i} \right]^2 = \max \left\{ \max_{i \in \bar{I}_\epsilon(1)} \left[\frac{2\sigma_i^2 i^{-\beta}}{k_i^2 \lambda_i} \right]^2, \max_{i \in \bar{I}_{\sigma\epsilon}(1)} \left[\frac{2c_0\epsilon_\sigma i^{-\beta}}{k_i^2 \lambda_i} \right]^2 \right\}.$$

Combining these results together, we obtain that on Ω_σ,

$$\begin{aligned}
&C\mathbb{E}_{\mu_0}\mathbb{E}(||\mu - \mu_0||^2 \mid Y, \hat{V}) \leq \epsilon^2 \sum_{i \in \bar{I}_\epsilon(2) \cup \bar{I}_\sigma(2)} \sigma_i^2 k_i^{-2} + \epsilon^2 c_0 \epsilon_\sigma \sum_{i \in I_\sigma(2) \cap \bar{I}_{\sigma\epsilon}(1/3)} k_i^{-2} \\
&+ \sum_{i \in I_\epsilon(2/3)} \lambda_i + \sum_{i \in I_\sigma(2) \cap I_{\sigma\epsilon}(1/3)} \lambda_i \\
&+ \epsilon^4 \max \left\{ \max_{i \in \bar{I}_\epsilon(1)} \left[\frac{\sigma_i^2 i^{-\beta}}{k_i^2 \lambda_i} \right]^2, \max_{i \in \bar{I}_{\sigma\epsilon}(1)} \left[\frac{\epsilon_\sigma i^{-\beta}}{k_i^2 \lambda_i} \right]^2 \right\} + \max_{i \in I_\epsilon(1) \cap I_{\sigma\epsilon}(1)} i^{-2\beta}
\end{aligned}$$

and hence this can be taken as ε^2, up to a constant. This bound is uniform in $\mu_0 \in S^\beta(A)$, and proves the rate stated in the theorem.

Note that

$$\begin{aligned}
I_\epsilon(a_1) \cap I_\sigma(a_2) &= \{i : \sigma_i^2/[\lambda_i k_i^2] \geq a_1 \epsilon^{-2}\} \cap \{i : \sigma_i^2 < a_2 c_0 \epsilon_\sigma\} \\
&\subseteq \{i : c_0\epsilon_\sigma/[\lambda_i k_i^2] \geq a_1/a_2 \epsilon^{-2}\} = I_{\sigma\epsilon}(a_1/a_2),
\end{aligned}$$

which also implies that $\bar{I}_{\sigma\epsilon}(a_1/a_2) \subseteq \bar{I}_\epsilon(a_1) \cup \bar{I}_\sigma(a_2)$, and hence

$$I_\sigma(2) \cap \bar{I}_{\sigma\epsilon}(1/3) \subseteq \bar{I}_\epsilon(2/3) \cap I_\sigma(2).$$

and

$$I_\epsilon(2) \cap \bar{I}_{\sigma\epsilon}(1/3) \subseteq I_\epsilon(2) \cap \bar{I}_\sigma(2/3).$$

If $I_\sigma(2/3) \subseteq I_\epsilon(2)$ then $\bar{I}_\epsilon(2) \cap \bar{I}_\sigma(2/3) = \bar{I}_\epsilon(2)$ and $\bar{I}_\epsilon(2) \cup I_\sigma(2/3) = \emptyset$, and the contraction rate can be written as

$$C\mathbb{E}_{\mu_0}\mathbb{E}(||\mu - \mu_0||^2 \mid Y, \hat{V}) \leq \epsilon^2 \sum_{i\in\bar{I}_\epsilon(2)} \sigma_i^2 k_i^{-2} + \sum_{i\in I_\epsilon(2)} \lambda_i$$
$$+\epsilon^4 \max_{i\in\bar{I}_\epsilon(1)} \left[\frac{\sigma_i^2 i^{-\beta}}{k_i^2\lambda_i}\right]^2 + \max_{i\in I_\epsilon(1)} i^{-2\beta}$$

which is almost the upper bound in the case V is known, except the summation in the first two sums is over $I_\epsilon(2)$ rather than $I_\epsilon(1)$. Here the sum of $\epsilon^2 c_0 \epsilon_\sigma k_i^{-2}$ over $I_\epsilon(2) \cap \bar{I}_{\sigma\epsilon}(1/3)$ is bounded by the sum of $\epsilon^2\lambda_i$ (up to a constant).

□

9. Discussion

We have consider the inverse problem with Gaussian errors in Hilbert spaces where the covariance operator is not constant and is known, and proved a theorem of contraction of the posterior distribution in the sequence space under a general setting. In particular, it is clear how the parameters of the prior affect contraction rate of the posterior distribution, and that in general we confirm the conclusion that undersmoothing leads to the consistency of the Bayesian estimator whereas oversmoothing does not. We also identified a setting where the posterior distribution can contract at the parametric rate, effectively leading to self-regularisation of the inverse problem.

We consider in detail a particular case of the covariance operator with when the eigenvalues of covariance operator decrease to 0 at a polynomial rate. We also derive minimax rates of convergence of estimators of the unknown signal under this model, and show the choice of the prior parameters that leads to this contraction rate of the posterior. Our simulation results confirm the theoretical conclusions.

One can argue that it is not realistic to assume the knowledge of the covariance operator, and it needs to be estimated in practice. Our results provide a benchmark of what can be expected in this case, and apply to the case of a plug-in estimator of variance. We discuss in which cases the rate of contraction of the posterior distribution is not affected whether the covariance operator is known exactly or up to an error.

Another potential question of interest is asymptotic equivalence of the Gaussian inverse problem (1) and a discretised version of n observations at some points x_i in the case the eigenvalues of the covariance operator of the noise tend to 0 in the sense of Le Cam, similarly to asymptotic equivalence of the white noise and nonparametric regression models.

Questions open for future research are existence of adaptive prior distribution that implies contraction of the posterior distribution at the minimax rate, i.e. when the prior distribution does not depend on smoothness of the unknown signal, as it was done by [Ray (2013)] and [Knapik et al.(2012)] under the white noise (homogeneous variance), and a joint Bayesian model of the signal and the variance function when the latter is unknown and its asymptotic behaviour.

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References

[Agapiou et al.(2013)] Agapiou, S., Larsson, S., Stuart, A.M. Posterior contraction rates for the Bayesian approach to linear ill-posed inverse problems. *Stochastic Processes and their Applications*, **2013**, *123*, 38283860.

[Agapiou and Mathe (2014)] Agapiou, S., and Mathe, P. Preconditioning the prior to overcome saturation in Bayesian inverse problems. *arXiv:1409.6496*, **2014**.

[Auranen at al.(2005)] Auranen, T., Nummenmaa, A., Hamalainen, M.S., Jaaskelainen, I., Lampinen, J., Vehtari, A., Sams, M. Bayesian Analysis of the Neuromagnetic Inverse Problem with lp Norm Priors. *NeuroImage*, **2005**, *26*, 870-884, 10.1016/j.neuroimage.2005.02.046.

[Belitser and Levit(1995)] Belitser, E. & Levit, B. On Minimax Filtering over Ellipsoids. *Math. Meth. Statist.*, **1995**, *3*, 259-273.

[Bochkina(2013)] Bochkina, N. Consistency of posterior distribution in generalized linear inverse problems. *Inverse Problems*, **2013**, *29*, 095010.

[Brown and Low (1996)] Brown, L. D., Low, M. G. Asymptotic Equivalence of Nonparametric Regression and White Noise. *Ann. Statist.*, **1996**, *24*, 23842398.

[Cai and Wang (2008)] T. Tony Cai and Lie Wang. Adaptive variance function estimation in heteroscedastic nonparametric regression. *Ann. Statist.*, **2008**, *36*, 2025-2054.

[Cavalier et al.(2002)] Cavalier, L., Golubev, G.K., Picard, D., Tsybakov, A.B. Oracle Inequalities for Inverse Problems. *Ann. Statist.* **2002**, *30*, 843874, 10.1214/AOS/1028674843.

[Cotter et al.(2009)] Cotter, S.L., Dashti, M., Robinson, J.C., Stuart, A.M. Bayesian Inverse Problems for Functions and Applications to Fluid Mechanics. *Inverse Probl.* **2009**, *25*, 10.1088/0266-5611/25/11/115008.

[Dashti et al.(2012)] Dashti, M., Harris, S., Stuart, A.M. Besov Priors for Bayesian Inverse Problems. *Inverse Probl. Imaging* **2012**, *6*, 183-200, 10.3934/ipi.2012.6.183.

[Efendiev et al.(2008)] Efendiev, Y., Datta Gupta, A., Hwang, K., Ma, X., Mallick, B. Bayesian Partition Models for Subsurface Characterization. In Large-Scale Inverse Problems and Quantification of Uncertainty; Wiley: New Jersey, U.S., 2008; pp 183 -199, 9780470685853.

[Engl(1996)] Engl, P.R. *Regularization of Inverse Problems*; Springer: Netherlands, 1996; 978-0-7923-6140-4.

[Florens and Simoni (2016)] Florens, J., Simoni, A. Regularizing priors for linear inverse problems. *Econometric Theory*, **2016**, *32(1)*, 71-121

[Ghosal and Ghosh (2000)] Ghosal, S., Ghosh, J.K., Convergence Rates of Posterior Distributions. *Ann. Statist.* **2000**, *28*, 500-531, 10.1214/aos/1016218228.

[Halmos(1974)] Halmos, P.R. The Volterra Operator. In *A Hilbert Space Problem Book*; Springer-Verlag: New York, U.S., 1974; 978-1-4615-9976-0.

[Kaipio et al.(1999)] Kaipio, J.P., Kolehmainen, V., Vauhkonen, M., Somersalo, E. Inverse Problems with Structural Prior Information. *Inverse Probl.* **1999**, *15*, 713-729.

[Kaipio and Fox (2011)] Kaipio, J.P., Fox, C. The Bayesian Framework for Inverse Problems in Heat Transfer. *Heat Transfer Eng.* **2011**, *32*, 718-753, 10.1080/01457632.2011.525137.

[Knapik et al.(2011)] Knapik, B., Van Der Vaart, A., Van Zanten, J.H. Bayesian Inverse Problems with Gaussian Priors. *Ann. Statist.* **2011**, *39*, 26262657, 10.1214/11-AOS920.

[Knapik et al.(2012)] Knapik, B.T., Szabo, B.T., Van der Vaart, A.W., Van Zanten, J.H. Bayes Procedures for Adaptive Inference in Inverse Problems for the White Noise Model. *ArXiv e-prints*, **2012**.

[Laurent and Massart (2000)] Laurent, B., Massart, P. Adaptive Estimation of a Quadratic Functional by Model Selection. *Ann. Statist.*, **2000**, *28*, Number 5, 1302-1338.

[Ray (2013)] Ray, K. Bayesian Inverse Problems with Non-Conjugate Priors. *Electron. J. Statist.*, **2013**, *7*, 2516-2549, 10.1214/13-EJS851.

[Rousseau and Szabo (2017)] Rousseau, J. and Szabo, B. Asymptotic behaviour of the empirical Bayes posteriors associated to maximum marginal likelihood estimator. *The Annals of Statistics*, 2017, Vol. 45, No. 2, 833865.

[Rover et al.(2007)] Rover, C., Meyer, R., Guidi, G. M., Vicere, A., Christensen, N. Coherent Bayesian Analysis of Inspiral Signals. *Class. Quantum Grav.*, **2007**, *24*, 607-615, 10.1088/0264-9381/24/19/S23.

[Spokoiny (2011)] Spokoiny, V. Parametric Estimation. Finite Sample Theory [Supplementary]. *Ann. Statist.*, **2011**, *40*, 10.1214/12-AOS1054SUPP.

[Tarantola(2013)] Tarantola, A. Popper, Bayes and the Inverse Problem. *Nat. Phys .*, **2006**, *2*, 492-494, 10.1038/nphys375.

[Tsybakov (2008)] Tsybakov, A.B. Introduction to Nonparametric Estimation. Springer, 2008.

[Voutilainen and Kaipio (2009)] Voutilainen, A., Kaipio, J.P. Model Reduction and Pollution Source Identification from Remote Sensing Data. *Inverse Probl. Imaging*, **2009**, *3*, 711-730, 10.3934/ipi.2009.3.711.

[Voxman (1981)] Voxman, W.L. The Dirichlet Test. In *Advanced Calculus: An Introduction to Modern Analysis*; Marcel Dekker: New York, U.S., 1981; 0-8247-6949-X.

[Wolpert and Ickstadt (2004)] Wolpert, R. L., Ickstadt, K. Reflecting Uncertainty in Inverse Problems: a Bayesian Solution using Levy Processes. *Inverse Probl.*, **2004**, *20*, 1759-1771, 10.1088/0266-5611/20/6/004.

Appendix A. Proofs when σ_i are polynomial

Proof: [Proof of Theorem 2] Here we give an alternative proof of the theorem (rather than using Theorem 1) following the proof of Theorem 4.1 in [Knapik et al.(2011)].

Similarly, the main tool we are going to use is Markov inequality:

$$\mathbb{P}(||\mu - \mu_0||_2 \geq M\varepsilon \mid Y) \leq M^{-2}\varepsilon^{-2}\mathbb{E}(||\mu - \mu_0||_2^2 \mid Y).$$

Hence, if we show that $\mathbb{E}_{\mu_0}\mathbb{E}(||\mu - \mu_0||_2^2 \mid Y) \leq C\varepsilon^2$ for some $C > 0$ independent of ε then ε is the rate of contraction of the posterior distribution.

Under assumptions 1, using Parseval's identity, we have that

$$\mathbb{E}(||\mu - \mu_0||^2 \mid Y) = \sum_i \mathbb{E}[(\mu_i - \mu_{0i})^2| \mid Y] = \sum_i [Var(\mu_i \mid Y) + (\mathbb{E}[\mu_i| \mid Y] - \mu_{0i})^2]$$

Taking the expected value with respect to the true distribution of the data and using the explicit form of the posterior distribution (5), we have

$$\begin{aligned}
\mathbb{E}_{\mu_0}[Var(\mu_i \mid Y)] + \mathbb{E}_{\mu_0}[(\mathbb{E}[\mu_i| \mid Y] - \mu_{0i})^2] &= \mathbb{E}_{\mu_0}\left[\frac{Y_i k_i \lambda_i}{\lambda_i k_i^2 + \epsilon^2\sigma_i^2} - \mu_{0i}\right]^2 + \frac{\sigma_i^2\lambda_i}{\epsilon^{-2}\lambda_i k_i^2 + \sigma_i^2} \\
&= \frac{\epsilon^2\sigma_i^4 k_i^2\lambda_i^2}{[\lambda_i k_i^2 + \epsilon^2\sigma_i^2]^2} + \mu_{0i}^2\left[\frac{\epsilon^2\sigma_i^2}{\lambda_i k_i^2 + \epsilon^2\sigma_i^2}\right]^2 + \frac{\sigma_i^2\lambda_i}{\epsilon^{-2}\lambda_i k_i^2 + \sigma_i^2} \\
&= S_1 + S_2 + S_3.
\end{aligned}$$

Defining $\tilde{k}_i := k_i/\sigma_i$, the three terms can be written as

$$\begin{aligned}
S_1 &= \epsilon^2 \sum_i \frac{\epsilon^{-4}\lambda_i^2 k_i^2 \sigma_i^4}{(\sigma_i^2 + \epsilon^{-2}\lambda_i k_i^2)^2} = \sum_i \frac{\epsilon^{-2}\lambda_i^2\tilde{k}_i^2}{(1 + \epsilon^{-2}\lambda_i\tilde{k}_i^2)^2} \\
S_2 &= \sum_i \frac{\mu_{0,i}^2\sigma_i^4}{(\sigma_i^2 + \epsilon^{-2}k_i^2\lambda_i)^2} = \sum_i \frac{\mu_{0,i}^2}{(1 + \epsilon^{-2}\tilde{k}_i^2\lambda_i)^2} \\
S_3 &= \sum_j \frac{\lambda_j\sigma_j^2}{\sigma_j^2 + \epsilon^{-2}k_j^2\lambda_j} = \sum_j \frac{\lambda_j}{1 + \epsilon^{-2}\tilde{k}_j^2\lambda_j}.
\end{aligned}$$

Using assumptions 2, 5 and 4, we have

$$\begin{aligned}
1 + \epsilon^{-2}\lambda_i C_1^{-2}C_2^{-2}i^{-2(p+\gamma)} \leq {} & 1 + \epsilon^{-2}\lambda_i\tilde{k}_i^2 \leq 1 + n\lambda_i C_1^2 C_2^2 i^{-2(p+\gamma)} \implies \\
C_l(1 + \epsilon^{-2}\lambda_i i^{-2\tilde{p}}) \leq {} & 1 + \epsilon^{-2}\lambda_i\tilde{k}_i^2 \leq C_u(1 + \epsilon^{-2}\lambda_i i^{-2\tilde{p}}) \implies \\
& 1 + \epsilon^{-2}\lambda_i\tilde{k}_i^2 \asymp 1 + \epsilon^{-2}\lambda_i i^{-2\tilde{p}}
\end{aligned}$$

where $\tilde{\boldsymbol{p}} = \boldsymbol{p} + \boldsymbol{\gamma}$, $C_l = \min(1, C_1^{-2}C_2^{-2})$ and $C_u = \max(1, C_1^2C_2^2)$.

Hence,

$$\begin{aligned}
S_2 &= \sum_i \frac{\mu_{0,i}^2}{(1 + \epsilon^{-2}\tilde{k}_i^2\lambda_i)^2} \asymp \sum_i \frac{\mu_{0,i}^2}{(1 + \epsilon^{-2}\tau_\epsilon^2 i^{-1-2\alpha-2\tilde{p}})^2} = ||\mu_0||_{S^\beta}^2 \sum_i \frac{\mu_{0,i}^2/||\mu_0||_{S^\beta}^2}{(1 + \epsilon^{-2}\tau_\epsilon^2 i^{-1-2\alpha-2\tilde{p}})^2} \\
&= ||\mu_0||_{S^\beta}^2 \sum_i \frac{\tilde{\mu}_{0,i}^2}{(1 + \epsilon^{-2}\tau_\epsilon^2 i^{-1-2\alpha-2\tilde{p}})^2} \leq ||\mu_0||_{S^\beta}^2 \sup_{||\tilde{\mu}||_{S^\beta} \leq 1} \sum_i \frac{\tilde{\mu}_i^2}{(1 + \epsilon^{-2}\tau_\epsilon^2 i^{-1-2\alpha-2\tilde{p}})^2} \\
&\asymp (\epsilon^{-2}\tau_\epsilon^2)^{-(\frac{2\beta}{1+2\alpha+2\tilde{p}} \wedge 2)}
\end{aligned}$$

where $\tilde{\mu}_{0,i}^2 := \mu_{0,i}^2/||\mu_0||_{S^\beta}^2$, using Lemma 8.1 from [Knapik et al.(2011)] with $q = \beta, t = 0, u = 1+2\alpha+2\tilde{p}, v = 2$ and $N = \epsilon^{-2}\tau_\epsilon^2$. In Lemmas 8.1 and 8.2 of [Knapik et al.(2011)] it is assumed that $u > 0$, which corresponds to our assumption $\gamma > -\frac{(1+2\alpha+2p)}{2}$.

Using Lemma 8.2 from [Knapik et al.(2011)], along with setting $S(i) = 1, q = -1/2, t = 2+4\alpha+2\tilde{p}, u = 1+2\alpha+2\tilde{p}, v = 2$ and $N = \epsilon^{-2}\tau_\epsilon^2$, we obtain

$$
\begin{aligned}
S_1 &= \sum_i \frac{\epsilon^{-2}\lambda_i^2\tilde{k}_i^2}{(1+\epsilon^{-2}\lambda_i\tilde{k}_i^2)^2} \asymp \epsilon^2 \sum_i \frac{\epsilon^{-4}\tau_\epsilon^4 i^{-2-4\alpha-2\tilde{p}}}{(1+\epsilon^{-2}\tau_\epsilon^2 i^{-1-2\alpha-2\tilde{p}})^2} \\
&\asymp \epsilon^2 \begin{cases} (\epsilon^{-2}\tau_\epsilon^2)^{-\frac{1+4\alpha+2\tilde{p}}{1+2\alpha+2\tilde{p}}}, & \text{if } \gamma > -p-1/2, \\ (\epsilon^{-2}\tau_\epsilon^2)^{-2} \displaystyle\sum_{i \le (\epsilon^{-2}\tau_\epsilon^2)^{(1/u)}} i^{-1}, & \text{if } \gamma = -p-1/2, \\ (\epsilon^{-2}\tau_\epsilon^2)^{-2}, & \text{if } \gamma < -p-1/2 \end{cases} \\
&\asymp \begin{cases} \tau_\epsilon^2(\epsilon^{-2}\tau_\epsilon^2)^{-\frac{2\alpha}{1+2\alpha+2\tilde{p}}}, & \text{if } \gamma > -p-1/2 \\ \epsilon^2 \log(\epsilon^{-2}\tau_\epsilon^2), & \text{if } \gamma = -p-1/2 \\ \epsilon^2, & \text{if } \gamma < -p-1/2 \end{cases}
\end{aligned}
$$

using $\sum_{i=1}^N 1/i = O(\log N)$ as $N \to \infty$.

Setting $S(i) = 1, q = -1/2, t = 1+2\alpha, u = 1+2\alpha+2\tilde{p}, v = 1$ and $N = \epsilon^{-2}\tau_\epsilon^2$ in Lemma 8.2 of [Knapik et al.(2011)], we obtain a bound for the last sum

$$
S_3 = \sum_i \frac{\lambda_j}{1+\epsilon^{-2}\tilde{k}_j^2\lambda_j} \asymp \sum_i \frac{\tau_\epsilon^2 i^{-1-2\alpha}}{1+\epsilon^{-2}\tau_\epsilon^2 i^{-1-2\alpha-2\tilde{p}}}
$$

$$
\sum_i \frac{i^{-1-2\alpha}}{1+\epsilon^{-2}\tau_\epsilon^2 i^{-1-2\alpha-2\tilde{p}}} \asymp \begin{cases} (\epsilon^{-2}\tau_\epsilon^2)^{-\frac{2\alpha}{1+2\alpha+2\tilde{p}}}, & \text{if } \gamma > -p-1/2. \\ \epsilon^2 \log(\epsilon^{-2}\tau_\epsilon^2), & \text{if } \gamma = -p-1/2 \\ \epsilon^2, & \text{if } \gamma < -p-1/2. \end{cases}
$$

Combining these results together, we can see that the

$$
\begin{aligned}
&\mathbb{E}_{\mu_0}\mathbb{P}(\{\mu : ||\mu-\mu_0|| \ge M\varepsilon|Y\}) \le \frac{1}{M^2\varepsilon^2}\mathbb{E}_{\mu_0}\mathbb{E}(||\mu-\mu_0||^2 \mid Y) \\
&\le C \begin{cases} \dfrac{1}{M^2\varepsilon^2}[||\mu_0||_{S^\beta}^2(\epsilon^{-2}\tau_\epsilon^2)^{-(\frac{2\beta}{1+2\alpha+2\tilde{p}}\wedge 2)} + \tau_\epsilon^2(\epsilon^{-2}\tau_\epsilon^2)^{-\frac{2\alpha}{1+2\alpha+2\tilde{p}}}], & \text{if } \gamma > -p-1/2. \\ \dfrac{1}{M^2\varepsilon^2}[||\mu_0||_{S^\beta}^2(\epsilon^{-2}\tau_\epsilon^2)^{-(\frac{2\beta}{1+2\alpha+2\tilde{p}}\wedge 2)} + \tau_\epsilon^2(\epsilon^{-2}\tau_\epsilon^2)^{-1}\log(\epsilon^{-2}\tau_\epsilon^2), & \text{if } \gamma = -p-1/2. \\ \dfrac{1}{M^2\varepsilon^2}[||\mu_0||_{S^\beta}^2(\epsilon^{-2}\tau_\epsilon^2)^{-(\frac{2\beta}{1+2\alpha+2\tilde{p}}\wedge 2)} + \tau_\epsilon^2(\epsilon^{-2}\tau_\epsilon^2)^{-1}], & \text{if } \gamma < -p-1/2. \end{cases}
\end{aligned}
$$

Hence, setting

$$
\varepsilon := \begin{cases} (\epsilon^{-2}\tau_\epsilon^2)^{-(\frac{\beta}{1+2\alpha+2(p+\gamma)}\wedge 1)} + \tau_\epsilon(\epsilon^{-2}\tau_\epsilon^2)^{-\frac{\alpha}{1+2\alpha+2(p+\gamma)}}, & \text{if } \gamma > -p-1/2. \\ (\epsilon^{-2}\tau_\epsilon^2)^{-(\frac{\beta}{1+2\alpha+2(p+\gamma)}\wedge 1)} + \epsilon[\log[(\epsilon^{-2}\tau_\epsilon^2)]]^{1/2}, & \text{if } \gamma = -p-1/2. \\ (\epsilon^{-2}\tau_\epsilon^2)^{-(\frac{\beta}{1+2\alpha+2(p+\gamma)}\wedge 1)} + \epsilon, & \text{if } \gamma < -p-1/2. \end{cases}
$$

ensures that the $\mathbb{E}_{\mu_0}\mathbb{P}(\{\mu : ||\mu-\mu_0|| \ge M\varepsilon|Y\}) \to 0$ for every $M \to \infty$.

$\square$

Proof: [Proof of Proposition 2]

We use the following lemma from [Laurent and Massart (2000)].

Lemma 1 *Let $(Y_1,\dots,Y_D)$ be i.i.d Gaussian variables, with mean 0 and variance 1. Let $a_1,\dots,a_D$ be non-negative. We set*

$$|a|_\infty = \sup_{i=1,\dots,D} |a_i|, \quad \textit{and} \quad |a|_2^2 = \sum_{i=1}^{D} a_i^2.$$

Let $Z = \sum_{i=1}^{D} a_i(Y_i^2-1)$. Then, the following inequalities hold for any positive x:

$$P(Z \geq 2|a|_2\sqrt{x} + 2|a|_\infty x) \leq e^{-x}$$
$$P(Z \leq -2|a|_2\sqrt{x}) \leq e^{-x}$$

Consequently, setting $D = m-1$, and $a_i = 1$ for all i, implies

$$Z = \sum_{i=1}^{m-1}(Y_i^2-1), \quad \text{and}$$
$$P(Z \geq 2|a|_2\sqrt{x} + 2|a|_\infty x) = P(Z \geq 2\sqrt{m-1}\sqrt{x} + 2x) \leq e^{-x}$$
$$P(Z \leq -2|a|_2\sqrt{x}) = P(Z \leq -2\sqrt{m-1}\sqrt{x}) \leq e^{-x}$$

Furthermore, for some $C > 0$,

$$\begin{aligned} 2\sqrt{m-1}\sqrt{x} + 2x = C \quad &\iff \sqrt{x} = -\frac{\sqrt{m-1}}{2} + \frac{\sqrt{m-1+2C}}{2} \\ &\implies x = \frac{(m-1)+C}{2} - \frac{1}{2}\sqrt{(m-1)[m-1+2C]} \qquad \text{(A.1)} \\ 2\sqrt{m-1}\sqrt{x} = C \quad &\iff \sqrt{x} = \frac{C}{2\sqrt{m-1}} \\ &\implies x = \frac{C^2}{4(m-1)} \qquad \text{(A.2)} \end{aligned}$$

Hence, for a fixed i,

$$\begin{aligned} P(|\hat\sigma_i^2 - \sigma_i^2| \geq c_0\epsilon_\sigma) &= P(\frac{m-1}{\sigma_i^2}\hat\sigma_i^2 - (m-1) \geq \frac{m-1}{\sigma_i^2}c_0\epsilon_\sigma) \\ &\quad + P(\frac{m-1}{\sigma_i^2}\hat\sigma_i^2 - (m-1) \leq -\frac{m-1}{\sigma_i^2}c_0\epsilon_\sigma), \\ &= P(Z \geq \frac{m-1}{\sigma_i^2}c_0\epsilon_\sigma) + P(Z \leq -\frac{m-1}{\sigma_i^2}c_0\epsilon_\sigma). \end{aligned}$$

Note,

$$\begin{aligned} P(|\hat\sigma_i^2 - \sigma_i^2| \leq c_0\epsilon_\sigma,\ i=1,\dots,M) &= 1 - P(\exists i \in \{1,\dots,M\} : |\hat\sigma_i^2 - \sigma_i^2| \geq c_0\epsilon_\sigma) \\ &\geq 1 - \sum_{i=1}^{M} P(|\hat\sigma_i^2 - \sigma_i^2| \geq c_0\epsilon_\sigma) \end{aligned}$$

Using Equations (A.1) and (A.2), Lemma 1 implies

$$\sum_{i=1}^{M} P(|\hat\sigma_i^2 - \sigma_i^2| \geq c_0\epsilon_\sigma) \leq \sum_{i=1}^{M}(e^{-x_{1,i}} + e^{-x_{2,i}}),$$

where, for $i = 1, \ldots, M$,

$$x_{1,i} = \frac{m-1}{2}(1 + \frac{c_0\epsilon_\sigma}{\sigma_i^2}) - \frac{m-1}{2}\sqrt{(1 + 2\frac{c_0\epsilon_\sigma}{\sigma_i^2})} \tag{A.3}$$

$$x_{2,i} = \frac{m-1}{4}(\frac{c_0\epsilon_\sigma}{\sigma_i^2})^2. \tag{A.4}$$

Note that for $x \geq 0$, $1 + x - \sqrt{1+2x} \geq 0$ and

$$1 + x - \sqrt{1+2x} = \frac{x^2}{1 + x + \sqrt{1+2x}} \geq \frac{x^2}{2(1+x)}. \tag{A.5}$$

Now, we show that $x_{1,i} \leq x_{2,i}$:

$$\begin{aligned} \frac{m-1}{2}\frac{c_0\epsilon_\sigma}{\sigma_i^2} + \frac{m-1}{2} - \frac{(m-1)}{2}\sqrt{1 + 2\frac{c_0\epsilon_\sigma}{\sigma_i^2}} &\leq \frac{m-1}{4}(\frac{c_0\epsilon_\sigma}{\sigma_i^2})^2 \\ \iff y^2/2 - y - 1 + \sqrt{1+2y} &\geq 0 \end{aligned}$$

Denoting $g(y) = y^2/2 - y - 1 + \sqrt{1+2y}$, we have

$$\begin{aligned} &g(0) \ = 0, \quad \text{and} \\ &g'(y) = y - 1 + \frac{1}{\sqrt{1+2y}} > 0 \iff y > 1 \quad \textit{or} \quad y \leq 1 \quad \textit{and} \quad 2y^2(3/2 - y) > 0 \end{aligned}$$

which does hold. Hence, $g(y) \geq 0$, for all $y \geq 0$, and therefore $x_{1,i} \leq x_{2,i}$.

Thus,

$$\begin{aligned} \sum_{i=1}^{M} P(|\hat{\sigma}_i^2 - \sigma_i^2| \geq c_0\epsilon_\sigma) \leq \sum_{i=1}^{M}(e^{-x_{1,i}} + e^{-x_{2,i}}) \leq \sum_{i=1}^{M}(2e^{-x_{1,i}}) \leq 2Me^{-\min_{i=1,\ldots M} x_{1,i}} \\ \leq 2Me^{-(m-1)(c_0\epsilon_\sigma/C_2)^2/6} \end{aligned}$$

since

$$\min_{i=1,\ldots M} x_{1,i} = x_{1,1} = \frac{m-1}{2}\left[\frac{c_0\epsilon_\sigma}{C_2} + 1 - \sqrt{1 + 2\frac{c_0\epsilon_\sigma}{C_2}}\right] \geq \frac{m-1}{4}\frac{(c_0\epsilon_\sigma/C_2)^2}{1 + c_0\epsilon_\sigma/C_2} \geq \frac{m-1}{6}(c_0\epsilon_\sigma/C_2)^2$$

using inequality (A.5) and small ϵ_σ (eg such that $c_0\epsilon_\sigma/C_2 \leq 1/2$).

For $i > M$, to have

$$P(|\hat{\sigma}_i^2 - \sigma_i^2| \leq c_0\epsilon_\sigma, \ i > M) = P(\sigma_i^2 \leq c_0\epsilon_\sigma, \ i > M) = 1,$$

we need $M > M_\sigma = \inf_M\{\sigma_i^2 \leq c_0\epsilon_\sigma, \forall i > M\}$.

□

Proof: [Proof of Corollary 3]

Note, the parameters of interest are m, ϵ_σ and M. Thus,

$$e^{-\frac{(m-1)}{6}(c_0\epsilon_\sigma/C_2)^2 + \log M} \to 0 \iff \frac{(m-1)}{6}(c_0\epsilon_\sigma/C_2)^2 - \log M \to \infty$$

However, since

$$\frac{(m-1)}{6}(c_0\epsilon_\sigma/C_2)^2 - \log M \geq Cm\epsilon_\sigma^2 - \log M$$

for some $C > 0$, it suffices to show

$$Cm\epsilon_\sigma^2 - \log M = Cm\epsilon_\sigma^2\left(1 - \frac{\log M}{Cm\epsilon_\sigma^2}\right) \to \infty \iff m\epsilon_\sigma^2 \to \infty, \quad \text{and} \quad \frac{\log M}{m\epsilon_\sigma^2} \to 0.$$

□

Appendix A.1. Proof for Polynomially decaying eigenvalues case

Proof: [Proof of Theorem 4]

Use Theorem 3 to derive the rate with $k_i \asymp i^{-p}$, $\sigma_i \asymp i^\gamma$, $\lambda_i \asymp \tau_\epsilon^2 i^{-2\alpha+1}$. First we investigate the sets

$$\begin{aligned}
I_\epsilon(a) &= \{i : \sigma_i^2/[\lambda_i k_i^2] > a\epsilon^{-2}\} = \{i : i > (a\tau_\epsilon^2\epsilon^{-2})^{1/(2\tilde{p}+2\alpha+1)}\},\\
I_\sigma(a) &= \{i : \sigma_i^2 < ac_0\epsilon_\sigma\} = \{i : i > (ac_0\epsilon_\sigma)^{1/(2\gamma)}\},\\
I_{\sigma\epsilon}(a) &= \{i : c_0\epsilon_\sigma/[\lambda_i k_i^2] > a\epsilon^{-2}\} = \{i : i > [a\tau_\epsilon^2\epsilon^{-2}\epsilon_\sigma^{-1}/c_0]^{1/(2p+2\alpha+1)}\}.
\end{aligned}$$

Denote

$$\begin{aligned}
i_\epsilon(a) &= (a\tau_\epsilon^2\epsilon^{-2})^{1/(2\tilde{p}+2\alpha+1)},\\
i_\sigma(a) &= (ac_0\epsilon_\sigma)^{1/(2\gamma)},\\
i_{\sigma\epsilon}(a) &= [a\tau_\epsilon^2\epsilon^{-2}\epsilon_\sigma^{-1}/c_0]^{1/(2p+2\alpha+1)}.
\end{aligned}$$

Then, the squared contraction rate given by Theorem 3 is

$$\begin{aligned}
\varepsilon_{plugin}^2 &= \epsilon^2 \sum_{i \le \max(i_\epsilon(2), i_\sigma(2))} i^{2\tilde{p}} + \epsilon^2 c_0\epsilon_\sigma \sum_{i > i_\sigma(2)\,\&\, i \le i_{\sigma\epsilon}(1/3)} i^{2p}\\
&+ \sum_{i > i_\epsilon(2/3)} \tau_\epsilon^2 i^{-2\alpha-1} + \sum_{i > \max(i_\sigma(2), i_{\sigma\epsilon}(1/3))} \tau_\epsilon^2 i^{-2\alpha-1}\\
&+ \epsilon^4 \max\left\{\max_{i \le i_\epsilon(1)} \left[\frac{\sigma_i^2 i^{-\beta}}{k_i^2\lambda_i}\right]^2, \max_{i \le i_{\sigma\epsilon}(1)} \left[\frac{\epsilon_\sigma i^{-\beta}}{k_i^2\lambda_i}\right]^2\right\} + \max_{i > \max(i_\epsilon(1), i_{\sigma\epsilon}(1))} i^{-2\beta}\\
&\asymp \epsilon^2 \max(i_\epsilon(2), i_\sigma(2))^{(2\tilde{p}+1)_+} + \epsilon^2 c_0\epsilon_\sigma (i_{\sigma\epsilon}(1/3)^{2p+1} - i_\sigma(2)^{2p+1}) |I(i_{\sigma\epsilon}(1/3) > i_\sigma(2))\\
&+ \tau_\epsilon^2[i_\epsilon(2/3)]^{-2\alpha} + \tau_\epsilon^2[\max(i_\sigma(2), i_{\sigma\epsilon}(1/3))]^{-2\alpha}\\
&+ \epsilon^4 \max\left\{[i_\epsilon(1)]^{2(2\tilde{p}+2\alpha+1-\beta)_+}, \epsilon_\sigma^2[i_{\sigma\epsilon}(1)]^{(2p+2\alpha+1-\beta)_+}\right\} + [\max(i_\epsilon(1), i_{\sigma\epsilon}(1))]^{-2\beta}\\
&\asymp \epsilon^2 \max((2\tau_\epsilon^2\epsilon^{-2})^{(2\tilde{p}+1)_+/(2\tilde{p}+2\alpha+1)}, (\epsilon_\sigma)^{(2\tilde{p}+1)_+/(2\gamma)})\\
&+ \epsilon^2\epsilon_\sigma \left([\tau_\epsilon^2\epsilon^{-2}\epsilon_\sigma^{-1}]^{(2p+1)/(2p+2\alpha+1)} - [(\epsilon_\sigma)^{(2p+1)/(2\gamma)}]\right)_+\\
&+ \tau_\epsilon^2(\tau_\epsilon^2\epsilon^{-2})^{-2\alpha/(2\tilde{p}+2\alpha+1)} + \tau_\epsilon^2[\max([\tau_\epsilon^2\epsilon^{-2}\epsilon_\sigma^{-1}]^{1/(2p+2\alpha+1)}, (\epsilon_\sigma)^{1/(2\gamma)})]^{-2\alpha}\\
&+ \epsilon^4 \max\left\{[(\tau_\epsilon^2\epsilon^{-2})^{1/(2\tilde{p}+2\alpha+1)}]^{2(2\tilde{p}+2\alpha+1-\beta)_+}, \epsilon_\sigma^2[[a\tau_\epsilon^2\epsilon^{-2}\epsilon_\sigma^{-1}/c_0]^{1/(2p+2\alpha+1)}]^{(2p+2\alpha+1-\beta)_+}\right\}\\
&+ [\max((\tau_\epsilon^2\epsilon^{-2})^{1/(2\tilde{p}+2\alpha+1)}, [\tau_\epsilon^2\epsilon^{-2}\epsilon_\sigma^{-1}]^{1/(2p+2\alpha+1)})]^{-2\beta}.
\end{aligned}$$

First consider the case $i_\epsilon(1) \le i_\sigma(1)$, i.e. if $\epsilon_\sigma \le (\tau_\epsilon^2\epsilon^{-2})^{2\gamma/(2\tilde{p}+2\alpha+1)}$. Below we consider $a = 1$ and omit this argument. In particular, this implies that $i_\epsilon \le i_{\sigma\epsilon} \le i_\sigma$. Then,

$$\varepsilon_{plugin}^2 \asymp \epsilon^2\epsilon_\sigma^{(2\tilde{p}+1)_+/(2\gamma)} + \tau_\epsilon^2(\tau_\epsilon^2\epsilon^{-2})^{-2\alpha/(2\tilde{p}+2\alpha+1)} + \tau_\epsilon^2[\epsilon_\sigma]^{-2\alpha/(2\gamma)} + [\tau_\epsilon^2\epsilon^{-2}\epsilon_\sigma^{-1}]^{-2\beta/(2p+2\alpha+1)}$$

$$
\begin{aligned}
&+\epsilon^4 \max\left\{\left(\tau_\epsilon^2\epsilon^{-2}\right)^{2(2\tilde{p}+2\alpha+1-\beta)_+/(2\tilde{p}+2\alpha+1)}, \epsilon_\sigma^2\left[\tau_\epsilon^2\epsilon^{-2}\epsilon_\sigma^{-1}\right]^{(2p+2\alpha+1-\beta)_+/(2p+2\alpha+1)}\right\}\\
&\leq \epsilon^2\left(\tau_\epsilon^2\epsilon^{-2}\right)^{(2\tilde{p}+1)_+/(2\tilde{p}+2\alpha+1)} + \tau_\epsilon^2\left(\tau_\epsilon^2\epsilon^{-2}\right)^{-2\alpha/(2\tilde{p}+2\alpha+1)} + \left(\tau_\epsilon^2\epsilon^{-2}\right)^{-2\beta/(2\tilde{p}+2\alpha+1)}\\
&+\epsilon^4\left(\tau_\epsilon^2\epsilon^{-2}\right)^{2(2\tilde{p}+2\alpha+1-\beta)_+/(2\tilde{p}+2\alpha+1)}\\
&\leq \left(\tau_\epsilon^2\right)^{(2\tilde{p}+1)_+/(2\tilde{p}+2\alpha+1)}\left(\epsilon^2\right)^{1-(2\tilde{p}+1)_+/(2\tilde{p}+2\alpha+1)} + \left(\tau_\epsilon^2\epsilon^{-2}\right)^{-2\beta/(2\tilde{p}+2\alpha+1)}\\
&+\epsilon^4\left(\tau_\epsilon^2\epsilon^{-2}\right)^{2(2\tilde{p}+2\alpha+1-\beta)_+/(2\tilde{p}+2\alpha+1)}.
\end{aligned}
$$

This coincides with the rate of contraction of μ when (σ_i) are known.

Now we consider large ϵ_σ i.e. $i_\epsilon(1) \geq i_\sigma(1)$.

Observe, $i_\epsilon \geq i_\sigma$ implies

$$
\begin{aligned}
&I_\sigma \cap \bar{I}_\epsilon = \{i : i_\sigma \leq i < i_\epsilon\}\\
&\bar{I}_\sigma \cap \bar{I}_\epsilon = \{i : i < i_\sigma,\ i < i_\epsilon\} = \bar{I}_\sigma\\
&\bar{I}_\sigma \cap I_\epsilon = \{i : i_\epsilon \leq i < i_\sigma\} = \emptyset
\end{aligned}
$$

Thus,

$$
\epsilon^{-2}\sum_{i\in\bar{I}_\sigma\cap I_\epsilon}\lambda_i^2 k_i^2\sigma_i^{-2} = 0
$$

$$
\epsilon^2\sum_{i\in\bar{I}_\epsilon\cap\bar{I}_\sigma}\sigma_i^2 k_i^{-2} = \epsilon^2\sum_{i\leq(i_\epsilon\wedge i_\sigma)} i^{2(p+\gamma)} \asymp \epsilon^2 i_\sigma^{(1+2(p+\gamma))_+}(\log i_\sigma)^{\mathbb{I}\{1+2(p+\gamma)=0\}}
$$

$$
\epsilon^2 c_0\epsilon_\sigma\sum_{i\in I_\sigma\cap\bar{I}_\epsilon} k_i^{-2} = \epsilon^2 c_0\epsilon_\sigma\sum_{i_\sigma\leq i<i_\epsilon} i^{2p} = \epsilon^2 c_0\epsilon_\sigma i_\epsilon^{1+2p}
$$

$$
\sum_{i\in I_\epsilon}\lambda_i = \tau_\epsilon^2\sum_{i_\epsilon\leq i} i^{-1-2\alpha} \asymp \tau_\epsilon^2 i_\epsilon^{-2\alpha}
$$

Specifically,

$$
\begin{aligned}
&\epsilon^4\max\left[\max_{i\in\bar{I}_\epsilon\cap I_\sigma}\left[\frac{c_0^2\epsilon_\sigma^2 i^{-2\beta}}{k_i^4\lambda_i^2}\right], \max_{i\in\bar{I}_\epsilon\cap\bar{I}_\sigma}\left[\frac{\sigma_i^4 i^{-2\beta}}{k_i^4\lambda_i^2}\right]\right]\\
&= \epsilon^4\tau_\epsilon^{-4}\max\left[\epsilon_\sigma^2\max_{i\in\bar{I}_\epsilon\cap I_\sigma}\left[c_0^2 i^{2(1+2\alpha+2p-\beta)}\right], \max_{i\in\bar{I}_\sigma}\left[i^{2(1+2\alpha+2(p+\gamma)-\beta)}\right]\right],
\end{aligned}
$$

where

$$
\begin{aligned}
&\epsilon^4\tau_\epsilon^{-4}\epsilon_\sigma^2\max_{i\in\bar{I}_\epsilon\cap I_\sigma}\left[c_0^2 i^{2(1+2\alpha+2p-\beta)}\right] = \epsilon^4\tau_\epsilon^{-4}\epsilon_\sigma^2\left[c_0^2 i_\epsilon^{2(1+2\alpha+2p-\beta)} \vee i_\sigma^{2(1+2\alpha+2p-\beta)}\right]\\
&= c_0^2 i_\epsilon^{-2\beta} \vee \epsilon^4\tau_\epsilon^{-4} i_\sigma^{2(1+2\alpha+2(p+\gamma)-\beta)},
\end{aligned}
$$

and

$$
\epsilon^4\tau_\epsilon^{-4}\max_{i\in\bar{I}_\sigma}\left[i^{2(1+2\alpha+2(p+\gamma)-\beta)}\right] = \epsilon^4\tau_\epsilon^{-4} \vee \epsilon^4\tau_\epsilon^{-4} i_\sigma^{2(1+2\alpha+2(p+\gamma)-\beta)}.
$$

Consequently, we obtain the following rates using Theorem 3:

$$\begin{aligned}\varepsilon^2_{plugin} &= \epsilon^2 i_\sigma^{(1+2(p+\gamma))_+}(\log i_\sigma)^{\mathbb{I}\{1+2(p+\gamma)=0\}} + \epsilon^2 c_0 \epsilon_\sigma i_\epsilon^{1+2p} + \tau_\epsilon^2 i_\epsilon^{-2\alpha} + i_\epsilon^{-2\beta} \\ &+ \left[\epsilon^4\tau_\epsilon^{-4} \vee \epsilon^4\tau_\epsilon^{-4} i_\sigma^{2(1+2\alpha+2(p+\gamma)-\beta)}\right]\end{aligned}$$

Note, using the definition of i_ϵ,

$$\begin{aligned}\epsilon^2 c_0\epsilon_\sigma i_\epsilon^{2p+1} &= \epsilon^2 c_0 \epsilon_\sigma i_\epsilon^{2p+1+2\alpha} i_\epsilon^{-2\alpha} = \epsilon^2 c_0\epsilon_\sigma(\epsilon^{-2}\tau_\epsilon^2\epsilon_\sigma^{-1}) i_\epsilon^{-2\alpha} = c_0\tau_\epsilon^2 i_\epsilon^{-2\alpha}. \\ \epsilon^2 i_\sigma^{(2(\gamma+p)+1)} &= \epsilon^2 c_0\epsilon_\sigma i_\sigma^{2p+1} \le \epsilon^2 c_0 \epsilon_\sigma i_\epsilon^{2p+1} = c_0\tau_\epsilon^2 i_\epsilon^{-2\alpha}\end{aligned}$$

Therefore,

$$\begin{aligned}\varepsilon^2_{plugin} &= \epsilon^2(\log i_\sigma)^{\mathbb{I}\{1+2(p+\gamma)=0\}} + \tau_\epsilon^2 i_\epsilon^{-2\alpha} + i_\epsilon^{-2\beta} + \left[\epsilon^4\tau_\epsilon^{-4} \vee \epsilon^4\tau_\epsilon^{-4} i_\sigma^{2(1+2\alpha+2(p+\gamma)-\beta)}\right] \\ &= \epsilon^2(\log\epsilon_\sigma^{-1})^{\mathbb{I}\{1+2(p+\gamma)=0\}} \\ &+ \tau_\epsilon^2[\epsilon_\sigma^{-1}\epsilon^{-2}\tau_\epsilon^2]^{-2\alpha/(1+2\alpha+2p)} + [\epsilon_\sigma^{-1}\epsilon^{-2}\tau_\epsilon^2]^{-2\beta/(1+2\alpha+2p)} \\ &+ \left[\epsilon^4\tau_\epsilon^{-4} \vee \epsilon^4\tau_\epsilon^{-4}\epsilon_\sigma^{\frac{1+2\alpha+2(p+\gamma)-\beta}{\gamma}}\right]\end{aligned}$$

□

Proof of Theorem 6

Define

$$q(\tau) = \sum_{i=1}^\infty \left[\frac{y_i^2}{k_i^2\lambda_{0,i}\tau + \epsilon^2\sigma_i^2} + \log(k_i^2\lambda_{0,i}\tau + \epsilon^2\sigma_i^2)\right]$$

Differentiating with respect to τ, we have

$$q'(\tau) = -\sum_{i=1}^\infty \frac{y_i^2 k_i^2\lambda_{0,i}}{[k_i^2\lambda_{0,i}\tau + \epsilon^2\sigma_i^2]^2} + \sum_{i=1}^\infty \frac{1}{\tau + \epsilon^2\sigma_i^2/(k_i^2\lambda_{0,i})}$$

and $\hat\tau$ satisfies $q'(\hat\tau) = 0$.

Denote $Z_i = \sigma_i^2/(k_i^2\lambda_{0,i})$ and choose α such that this sequence increases. Define

$$i_\epsilon = \max\{i:\ k_i^2\lambda_{0,i}\tau \ge \epsilon^2\sigma_i^2\} = \max\{i:\ Z_i \le \tau\epsilon^{-2}\}.$$

Note that for $i > i_\epsilon$, $Z_i > \tau\epsilon^{-2}$.

Recall that $y_i = \mu_{0,i}k_i + \epsilon\sigma_i\xi_i$ where $\xi_i \sim N(0,1)$ iid, $i = 1, 2, \ldots$. Then,

$$\begin{aligned}q'(\tau) &= -\sum_{i=1}^\infty \frac{[\mu_{0,i}^2 k_i^2 + \epsilon^2\sigma_i^2 + 2\mu_{0,i}^2 k_i\epsilon\sigma_i\xi_i + \epsilon^2\sigma_i^2(\xi_i^2-1)]k_i^2\lambda_{0,i}}{[k_i^2\lambda_{0,i}\tau + \epsilon^2\sigma_i^2]^2} + \sum_{i=1}^\infty \frac{1}{\tau + \epsilon^2\sigma_i^2/(k_i^2\lambda_{0,i})} \\ &= -\sum_{i=1}^\infty \frac{\mu_{0,i}^2/\lambda_{0,i} + 2\mu_{0,i}\sqrt{Z_i/\lambda_{0,i}}\epsilon\xi_i + \epsilon^2 Z_i(\xi_i^2-1) - \tau}{[\tau + \epsilon^2 Z_i]^2}\end{aligned}$$

Note that $\mu_{0,i}^2/\lambda_{0,i} = i^{\alpha+1/2}\mu_{0,i}^2$. Using similar technique as in the proof of Theorem 1, we have

$$q'(\tau) \asymp -\sum_{i\le i_\epsilon} \frac{\mu_{0,i}^2/\lambda_{0,i} + 2\mu_{0,i}\sqrt{Z_i/\lambda_{0,i}}\epsilon\xi_i + \epsilon^2 Z_i(\xi_i^2-1) - \tau}{\tau^2}$$

$$- \sum_{i>i_\epsilon} \frac{\mu_{0,i}^2/\lambda_{0,i} + 2\mu_{0,i}\sqrt{Z_i/\lambda_{0,i}}\epsilon\xi_i + \epsilon^2 Z_i(\xi_i^2 - 1) - \tau}{[\epsilon^2 Z_i]^2}$$

$$\asymp i_\epsilon/\tau - \tau^{-2} \sum_{i\le i_\epsilon} \left[\mu_{0,i}^2/\lambda_{0,i} + 2\mu_{0,i}\sqrt{Z_i/\lambda_{0,i}}\epsilon\xi_i + \epsilon^2 Z_i(\xi_i^2 - 1)\right]$$

$$- \epsilon^{-4} \sum_{i>i_\epsilon} [Z_i^{-2}\mu_{0,i}^2/\lambda_{0,i} + 2\mu_{0,i}\sqrt{1/\lambda_{0,i}}Z_i^{-3/2}\epsilon\xi_i + \epsilon^2 Z_i^{-1}(\xi_i^2 - 1)] + \tau\epsilon^{-4} \sum_{i>i_\epsilon} Z_i^{-2}$$

Denote the random term

$$W = \tau^{-2} \sum_{i\le i_\epsilon} \left[2\mu_{0,i}\sqrt{Z_i/\lambda_{0,i}}\epsilon\xi_i + \epsilon^2 Z_i(\xi_i^2 - 1)\right] + \epsilon^{-4} \sum_{i>i_\epsilon} [2\mu_{0,i}\sqrt{1/\lambda_{0,i}}Z_i^{-3/2}\epsilon\xi_i + \epsilon^2 Z_i^{-1}(\xi_i^2 - 1)].$$

Then

$$q'(\tau) \asymp i_\epsilon/\tau - \tau^{-2} \sum_{i\le i_\epsilon} \mu_{0,i}^2/\lambda_{0,i} - \epsilon^{-4} \sum_{i>i_\epsilon} Z_i^{-2}\mu_{0,i}^2/\lambda_{0,i} + \tau\epsilon^{-4} \sum_{i>i_\epsilon} Z_i^{-2} + W$$

In the polynomial case, $Z_i \asymp i^{2\tilde{p}+2\alpha+1} = i^{a+1}$ and $i_\epsilon \asymp (\tau\epsilon^{-2})^{1/(1+a)}$, assuming $a + 1 > 0$, where $a = 2(\tilde{p} + \alpha)$; denote also $\Delta = \alpha - \beta$.

Plugging in the values of Z_i and $\lambda_{0,i}$, we have

$$\begin{aligned}
q'(\tau) &\asymp i_\epsilon/\tau - \tau^{-2} \sum_{i\le i_\epsilon} i^{2\beta}\mu_{0,i}^2 i^{2\Delta+1} - \epsilon^{-4} \sum_{i>i_\epsilon} i^{2\beta}\mu_{0,i}^2 i^{2\Delta-2a-1} + \tau\epsilon^{-4} \sum_{i>i_\epsilon} i^{-2a-2} + W \\
&\asymp i_\epsilon/\tau - \tau^{-2}||\mu_{0,i}||_{S^\beta}^2 i_\epsilon^{(2\Delta+1)_+} - \epsilon^{-4}||\mu_{0,i}||_{S^\beta}^2 i_\epsilon^{2\Delta-2a-1} + \tau\epsilon^{-4} i_\epsilon^{-(2a+1)} + W \\
&\asymp (\tau\epsilon^{-2})^{-a/(1+a)}\epsilon^{-2} - \epsilon^{-4}(\tau\epsilon^{-2})^{(2\Delta+1)_+/(1+a)-2} + W \\
&\asymp \tau^{-1}(\tau\epsilon^{-2})^{1/(1+a)} - \tau^{-2}(\tau\epsilon^{-2})^{(2\Delta+1)_+/(1+a)} + W
\end{aligned}$$

assuming $\Delta - a - 1/2 \le 0$ and $2a + 1 > 0$.

Now consider the random term. We use the following inequality from [Laurent and Massart (2000)]: if $\psi(u) = \log E \exp(uZ) \le \frac{vu^2}{2(1-cu)}$ for some $v > 0$, $c \ge 0$ then $P(Z > cx + \sqrt{2vx}) \le e^{-x}$. It is easy to show that for $\xi \sim N(0, 1)$ and $t < 1/2$,

$$\log E \exp(t[a\xi + (\xi^2 - 1)]) = \frac{a^2 t^2}{4(1/2 - t)} - 0.5 \log(1 - 2t) \le \frac{(1 + a^2/2)t^2}{1 - 2t}.$$

For the random term W defined above,

$$\begin{aligned}
\log E \exp(uW) &= \sum_{i\le i_\epsilon} \log E \exp\left[u\tau^{-2}\epsilon^2 Z_i(2\mu_{0,i}\sqrt{1/\lambda_{0,i}}Z_i^{-1/2}\epsilon^{-1}\xi_i + (\xi_i^2 - 1))\right] \\
&\quad + \sum_{i>i_\epsilon} \log E \exp\left[u\epsilon^{-2} Z_i^{-1}[2\mu_{0,i}\sqrt{1/\lambda_{0,i}}Z_i^{-1/2}\epsilon^{-1}\xi_i + (\xi_i^2 - 1)]\right] \\
&\le [u\tau^{-2}\epsilon^2]^2 \sum_{i\le i_\epsilon} \frac{(1 + a_i^2/2)Z_i^2}{1 - 2u\tau^{-2}\epsilon^2 Z_i} + [u\epsilon^{-2}]^2 \sum_{i>i_\epsilon} \frac{(1 + a_i^2/2)Z_i^{-2}}{1 - 2u\epsilon^{-2} Z_i^{-1}}
\end{aligned}$$

using the above expression with $a = a_i = 2\mu_{0,i}\sqrt{1/\lambda_{0,i}}Z_i^{-1/2}\epsilon^{-1}$. In the polynomial case,

$$\max_{i\le i_\epsilon} Z_i = i_\epsilon^{1+a} = \tau\epsilon^{-2}, \quad \max_{i\ge i_\epsilon} Z_i^{-1} = i_\epsilon^{-(1+a)} = \tau^{-1}\epsilon^2.$$

Hence,

$$\log E\exp(uW) \le \frac{u^2}{1-2u\tau^{-1}}\left[\tau^{-4}\epsilon^4\sum_{i\le i_\epsilon}(1+a_i^2/2)Z_i^2+\epsilon^{-4}\sum_{i>i_\epsilon}(1+a_i^2/2)Z_i^{-2}\right].$$

Also,

$$\begin{aligned}\sum_{i\le i_\epsilon}(1+a_i^2/2)Z_i^2 &= \sum_{i\le i_\epsilon}(1+2\mu_{0,i}^2/\lambda_{0,i}Z_i^{-1}\epsilon^{-2})Z_i^2 \asymp \sum_{i\le i_\epsilon} i^{2(1+a)}+\epsilon^{-2}\sum_{i\le i_\epsilon} i^{2\beta}\mu_{0,i}^2 i^{2\Delta+a+2}\\ &\asymp i_\epsilon^{3+2a}+\epsilon^{-2}i_\epsilon^{(2\Delta+a+2)_+}\end{aligned}$$

and

$$\begin{aligned}\sum_{i>i_\epsilon}(1+a_i^2/2)Z_i^{-2} &= \sum_{i>i_\epsilon}(1+2\mu_{0,i}^2/\lambda_{0,i}Z_i^{-1}\epsilon^{-2})Z_i^{-2}\\ &\asymp \sum_{i>i_\epsilon} i^{-2(1+a)}+\epsilon^{-2}\sum_{i>i_\epsilon} i^{2\beta}\mu_{0,i}^2 i^{2\Delta+1-3(a+1)}\\ &\asymp i_\epsilon^{-1-2a}+\epsilon^{-2}i_\epsilon^{2\Delta+1-3(a+1)}\end{aligned}$$

provided $2\Delta+1-3(a+1)\le 0$.

Therefore, for $u<\tau/2$ and $\Delta\le 3a/2+1$,

$$\begin{aligned}\log E\exp(uW) &\le C\frac{u^2\tau^{-2}}{1-2u\tau^{-1}}\left[(\tau\epsilon^{-2})^{1/(1+a)}+\epsilon^2(\tau\epsilon^{-2})^{(2\Delta+1+a+1)_+/(1+a)}\right.\\ &\quad\left.+(\tau\epsilon^{-2})^{1/(1+a)}+\tau^{-1}(\tau\epsilon^{-2})^{(2\Delta+1)/(a+1)}\right]\\ &\le C\frac{u^2\tau^{-2}}{1-2u\tau^{-1}}\left[(\tau\epsilon^{-2})^{1/(1+a)}+\epsilon^2(\tau\epsilon^{-2})^{(2\Delta+1+a+1)_+/(1+a)}\right].\end{aligned}$$

This implies that for $u\in(0,\tau/2)$,

$$P(W>C\tau^{-1}x+C\tau^{-1}\sqrt{x[(\tau\epsilon^{-2})^{1/(1+a)}+\epsilon^2(\tau\epsilon^{-2})^{(2\Delta+1+a+1)_+/(1+a)}]})\le e^{-x},$$

and for $u<0$ and $Z=-W$, we can take $c=0$ and hence

$$P(W<-C\tau^{-1}\sqrt{x[(\tau\epsilon^{-2})^{1/(1+a)}+\epsilon^2(\tau\epsilon^{-2})^{(2\Delta+1+a+1)_+/(1+a)}]})\le e^{-x}.$$

Denote $\tilde W=W\tau$ and $D=\sqrt{(\tau\epsilon^{-2})^{1/(1+a)}+\epsilon^2(\tau\epsilon^{-2})^{(2\Delta+1+a+1)_+/(1+a)}}$ and

$$\Omega_x=\{\tilde W:\ \tilde W<-CD\sqrt{x}\ \&\ \tilde W>Cx+C\sqrt{x}D\}.$$

If $\Delta\le a+1/2$, $\Delta\le 3a/2+1$, $2a+1>0$, then

$$q'(\tau)\asymp\tau^{-1}[(\tau\epsilon^{-2})^{1/(1+a)}-\tau^{-1}(\tau\epsilon^{-2})^{(2\Delta+1)_+/(1+a)}+\tilde W].$$

Note that the upper and lower bounds on $\tilde W$, D (up to $\sqrt{x}$) are smaller than other terms, if $\tau\epsilon^{-2}$ is large and ϵ is small:

$$\frac{\tau^{-1}(\tau\epsilon^{-2})^{(2\Delta+1)_+/(1+a)}}{\epsilon(\tau\epsilon^{-2})^{0.5(2\Delta+1+a+1)_+/(1+a)}}=\tau^{-1}\epsilon^{-1}(\tau\epsilon^{-2})^{-0.5(2\Delta+1+a+1)_+/(1+a)+(2\Delta+1)_+/(1+a)}.$$

If $2\Delta+1>0$ (and we know $a+1>0$), the ratio is

$$\tau^{-1}\epsilon^{-1}(\tau\epsilon^{-2})^{[0.5(2\Delta+1)-0.5(a+1)]/(1+a)}=\epsilon^{-3}(\tau\epsilon^{-2})^{0.5[(2\Delta+1)-3(a+1)]/(1+a)}\to\infty$$

as $(2\Delta+1)-3(a+1)>0$.

Comparing to the other term in the case $2\Delta+1\le 0$,

$$\frac{(\tau\epsilon^{-2})^{1/(1+a)}}{\epsilon^{-1}(\tau\epsilon^{-2})^{-0.5}}=\tau^{0.5}(\tau\epsilon^{-2})^{1/(1+a)}\to\infty.$$

The other term is

$$\frac{(\tau\epsilon^{-2})^{1/(1+a)}}{(\tau\epsilon^{-2})^{0.5/(1+a)}}=(\tau\epsilon^{-2})^{0.5/(1+a)}\to\infty.$$

Hence,

$$q'(\tau)\asymp\tau^{-1}[(\tau\epsilon^{-2})^{1/(1+a)}(1+o_P(1))-\epsilon^{-2}(\tau\epsilon^{-2})^{(2\Delta+1)_+/(1+a)-1}(1+o_P(1))]$$

implying that $\hat\tau$ satisfies

$$0=q'(\hat\tau)\asymp\hat\tau^{-1}[(\hat\tau\epsilon^{-2})^{1/(1+a)}(1+o_P(1))-\epsilon^{-2}(\hat\tau\epsilon^{-2})^{(2\Delta+1)_+/(1+a)-1}(1+o_P(1))]$$

i.e.

$$\hat\tau^{2+a-(2\Delta+1)_+}=\epsilon^{2[1-(2\Delta+1)_+]}(1+o_P(1)). \tag{A.6}$$

First consider the case $2\Delta+1\le 0$. Then, $\hat\tau\asymp\epsilon^{2/(2+a)}(1+o_P(1))=\epsilon^{1/(1+\tilde p+\alpha)}(1+o_P(1))$. In the case $2\Delta+1>0$, $(\hat\tau\epsilon^{-2})^{1+a-2\Delta}\asymp\epsilon^{-2(1+a)}(1+o_P(1))$, i.e. $\hat\tau\asymp\epsilon^{-4(\alpha-\beta)/(1+2\tilde p+2\beta)}(1+o_P(1))$. This proves (16).

Note that these conditions were derived under assumptions $\Delta\le a+1/2$ ($\beta+\alpha+2\tilde p+1/2\ge 0$), $2a+1>0$; assumption $\Delta\le 3a/2+1$ follows from these two. To avoid a constraint on β, we can take $\alpha>0$ such that $\alpha+2\tilde p+1/2\ge 0$.

Now we verify that $\hat\tau$ corresponds to the minimum of q.

$$\begin{aligned}
q''(\tau)&=2\sum_{i=1}^{\infty}\frac{y_i^2k_i^4\lambda_{0,i}^2}{[k_i^2\lambda_{0,i}\tau+\epsilon^2\sigma_i^2]^3}-\sum_{i=1}^{\infty}\frac{1}{[\tau+\epsilon^2\sigma_i^2/(k_i^2\lambda_{0,i})]^2}\\
&=\sum_{i=1}^{\infty}\frac{2\mu_{0,i}^2/\lambda_i+\epsilon^2Z_i+2\epsilon^2Z_i(\xi_i^2-1)+4\epsilon\xi_i\sqrt{Z_i}\mu_{0,i}/\sqrt{\lambda_i}-\tau}{[\tau+\epsilon^2Z_i^2]^3}\\
&\asymp\tau^{-3}\sum_{i\le i_\epsilon}[2\mu_{0,i}^2/\lambda_i+\epsilon^2Z_i-\tau]+\epsilon^{-6}\sum_{i>i_\epsilon}[2\mu_{0,i}^2/(\lambda_iZ_i^3)+\epsilon^2Z_i^{-2}-\tau/Z_i^3]\\
&\quad+\tau^{-1}W_{\le}+\epsilon^{-6}\sum_{i>i_\epsilon}2Z_i^{-1}[\epsilon^2Z_i^{-2}(\xi_i^2-1)+2\epsilon\xi_iZ_i^{-3/2}\mu_{0,i}/\sqrt{\lambda_i}]
\end{aligned}$$

The sums of the fixed terms are equal to, up to a constant,

$$\begin{aligned}
&\tau^{-3}i_\epsilon^{(2(\alpha-\beta)+1)_+}+\epsilon^2\tau^{-3}i_\epsilon^{2\tilde p+2\alpha+2}-\tau^{-2}i_\epsilon+\epsilon^{-6}i_\epsilon^{-2\beta-2-6\tilde p-4\alpha)}\\
&+\epsilon^{-4}i_\epsilon^{-2(2\tilde p+2\alpha+1)+1}-\tau\epsilon^{-6}i_\epsilon^{-3(2\tilde p+2\alpha+1)+1}\\
&=\tau^{-3}(\tau\epsilon^{-2})^{(2(\alpha-\beta)+1)_+/(1+a)}+\tau^{-2}(\tau\epsilon^{-2})^{1/(1+a)}-\tau^{-2}(\tau\epsilon^{-2})^{1/(a+1)}\\
&+\tau^{-3}(\tau\epsilon^{-2})^{(-2\beta+1+2\alpha)/(1+a)}
\end{aligned}$$

provided $\beta+1+3\tilde p+2\alpha\ge 0$ (which holds as $\beta+\alpha+2\tilde p+1/2\ge 0$ and $\alpha+\tilde p+1/2>0$), and at $\hat\tau$, it is positive:

$$\tau^{-3}(\tau\epsilon^{-2})^{(2(\alpha-\beta)+1)_+/(1+a)}+\tau^{-3}(\tau\epsilon^{-2})^{(-2\beta+1+2\alpha)/(1+a)}.$$

The second part of the random term can be bounded in absolute value by

$$\epsilon^{-6}|\sum_{i>i_\epsilon} Z_i^{-1}[\epsilon^2 Z_i^{-2}(\xi_i^2-1)+2\epsilon\xi_i Z_i^{-3/2}\mu_{0,i}/\sqrt{\lambda_i}]| \leq \epsilon^{-4}\tau^{-1}|\sum_{i>i_\epsilon}[\epsilon^2 Z_i^{-2}(\xi_i^2-1)+2\epsilon\xi_i Z_i^{-3/2}\mu_{0,i}/\sqrt{\lambda_i}]|$$

and hence the random term, in absolute value, is bounded above by

$$\tau^{-1}|W_{\leq}| + \tau^{-1}|W_{>}|$$

where the original random term, W, can be written as $W = W_{\leq} + W_{>}$, and hence, using similar technique, can be shown to be of the smaller order compared to the non-random term. Therefore, the second derivative is positive and $\hat{\tau}$ corresponds to the minimum.

Note that $\hat{\tau}\epsilon^{-2} \asymp (\epsilon^{-2})^{(1+a)/(2+a)}(1+o_P(1)) \to \infty$ as $\epsilon \to 0$ if $2\Delta+1 \leq 0$, and in the case $2\Delta+1 > 0$, $\hat{\tau}\epsilon^{-2} \asymp \epsilon^{-2[2\alpha+1+2\tilde{p}]/(1+2\tilde{p}+2\beta)}(1+o_P(1)) \to \infty$. Plugging the expressions for $\hat{\tau}$ in the rate stated in Theorem 2 and simplifying where possible, we obtain

$$\varepsilon = \epsilon^{2[\beta\wedge(\alpha+1/2)\wedge(1+2\alpha+2\tilde{p})]/(1+2\tilde{p}+2(\beta\wedge(\alpha+1/2)))} + \epsilon[\log\epsilon^{-1}]^{0.5I(p+\gamma=-1/2)}.$$

Under condition $\alpha + 2\tilde{p} + 1/2 \geq 0$, $(\alpha+1/2)\wedge(1+2\alpha+2\tilde{p}) = \alpha+1/2$ which simplifies the above expression.

This rate is optimal for $p+\gamma > -1/2$ if $\beta \leq \min(\alpha+1/2, 2\alpha+1+2\tilde{p})$, and it is optimal for $p+\gamma \leq -1/2$ if

$$2\beta\wedge(\alpha+1/2)\wedge(1+2\alpha+2\tilde{p}) \geq 1+2\tilde{p}+2(\beta\wedge(\alpha+1/2)).$$

First consider the case $\beta \geq (\alpha+1/2)$. Then, the inequality is equivalent to

$$2(\alpha+1/2)\wedge(1+2\alpha+2\tilde{p}) \geq 2+2\tilde{p}+2\alpha$$

which holds if $\alpha+\tilde{p} \geq 0$. If $\beta < \alpha+1/2$, then, the inequality is equivalent to

$$2\beta \geq 1+2\tilde{p}+2\beta$$

which holds.

Therefore, this rate is optimal for $p+\gamma > -1/2$ if $\beta \leq \min(\alpha+1/2, 2\alpha+1+2\tilde{p})$, and it is optimal for $p+\gamma \leq -1/2$ if $\alpha+\tilde{p} \geq 0$.

To summarise, for the posterior distribution with plugged in $\hat{\tau}$ to be optimal, we need $\alpha+\tilde{p} \geq 0$, and for $\beta \leq B_0$, take $\alpha \geq \max(B_0/2-(1/2+p+\gamma), B_0-1/2, -(p+\gamma))$. □